

EQUIVARIANT DEGENERATIONS OF SPHERICAL MODULES: PART II

STAVROS ARGYRIOS PAPADAKIS AND BART VAN STEIRTEGHEM

ABSTRACT. We determine, under a certain assumption, the Alexeev–Brion moduli scheme M_S of affine spherical G -varieties with a prescribed weight monoid S . In [PVS12] we showed that if G is a connected complex reductive group of type A and S is the weight monoid of a spherical G -module, then M_S is an affine space. Here we prove that this remains true without any restriction on the type of G .

1. INTRODUCTION AND STATEMENT OF RESULTS

A natural invariant of an affine variety X equipped with an action of a complex connected reductive group G is its **weight monoid** $\mathcal{S}(X)$. It is the set of (isomorphism classes of) irreducible representations of G that occur in the ring of regular functions $\mathbb{C}[X]$. If every irreducible representation occurs at most once in this ring, then X is called **multiplicity-free**. If, in addition, X is normal, then it is an **affine spherical variety**. For multiplicity-free varieties, the weight monoid completely describes the structure of $\mathbb{C}[X]$ as a representation of G . Knop’s Conjecture, proved by Losev in [Los09], asserts that if X is smooth and multiplicity-free, then $\mathcal{S}(X)$ uniquely determines X . This is no longer true without the smoothness assumption. A moduli scheme introduced by V. Alexeev and M. Brion [AB05] brings geometry to the natural question, “to what extent does $\mathcal{S}(X)$ determine X as a variety?”

To describe the moduli scheme, following [Bri13, Section 4.3], we introduce some more notation. Let B be a Borel subgroup of G . Then $B = TU$ where T is a maximal torus of G and U is the unipotent radical of B . Let Λ^+ be the monoid of dominant weights in the weight lattice Λ . Recall that by highest weight theory, the elements of Λ^+ are in bijection with the isomorphism classes of irreducible representations of G . Under this identification, the weight monoid $\mathcal{S}(X)$ of a multiplicity-free affine G -variety X is a finitely generated submonoid of Λ^+ . Now, given a finitely generated submonoid S of Λ^+ , define the following G -module: $V(S) := \bigoplus_{\lambda \in S} V(\lambda)$. By identifying it with the semigroup algebra $\mathbb{C}[S]$, we equip the space of highest weight vectors $V(S)^U$ with a T -multiplication law. The moduli scheme M_S introduced in [AB05] parametrizes the G -multiplication laws on $V(S)$ that extend the chosen T -multiplication law on the subspace $V(S)^U$. We will sometimes write M_S^G instead of M_S when we need to specify the group under consideration. Alexeev and Brion showed that M_S is an affine scheme of finite type over \mathbb{C} .

In more geometric language, the moduli scheme M_S parametrizes pairs (X, φ) where X is a multiplicity-free affine G -variety with weight monoid S and φ is a T -equivariant map $\mathrm{Spec}(\mathbb{C}[X]^U) \rightarrow \mathrm{Spec}(\mathbb{C}[S])$. Alexeev and Brion equipped M_S with a natural action of the ‘adjoint torus’ $T_{\mathrm{ad}} := T/Z(G)$, where $Z(G)$ is the center of G . They proved that the orbits correspond to isomorphism classes of multiplicity-free affine varieties with weight monoid S , and that there is a unique closed T_{ad} -orbit, which is a fixed point denoted X_0 .

Finally, they showed that if X is an affine multiplicity-free variety with weight monoid \mathcal{S} , and we think of X as a closed point on $M_{\mathcal{S}}$, then the closure of the orbit $T_{\text{ad}} \cdot X \subseteq M_{\mathcal{S}}$ has coordinate ring $\mathbb{C}[\Sigma_X]$, where Σ_X is the so-called **root monoid** of X :

$$\Sigma_X := \langle \lambda + \mu - \nu : \lambda, \mu, \nu \in \Lambda^+ \text{ such that } \langle \mathbb{C}[X]_{(\lambda)} \cdot \mathbb{C}[X]_{(\mu)} \rangle_{\mathbb{C}} \cap \mathbb{C}[X]_{(\nu)} \neq 0 \rangle_{\mathbb{N}}.$$

Here $\mathbb{C}[X]_{(\lambda)}$ is the isotypic component of $\mathbb{C}[X]$ of type $\lambda \in \Lambda^+$.

1.1. Main results. In [PVS12] we proved that if \mathcal{S} is the weight monoid of a spherical G -module, where G is of type A, then $M_{\mathcal{S}}$ is an affine space. Here we extend this result to weight monoids of spherical G -modules for arbitrary connected reductive groups G . That is, we here prove the following.

Theorem 1.1. *Assume W is a spherical G -module, where G is a connected reductive algebraic group. Let \mathcal{S} be the weight monoid of W and let d_W be the rank of the group $\mathbb{Z}\Sigma_W$ generated by the root monoid Σ_W of W . Then*

- (a) Σ_W is a freely generated monoid; and
- (b) the T_{ad} -scheme $M_{\mathcal{S}}$ is T_{ad} -equivariantly isomorphic to the T_{ad} -module with weight monoid Σ_W . In particular, the scheme $M_{\mathcal{S}}$ is isomorphic to the affine space \mathbb{A}^{d_W} , hence it is irreducible and smooth.

We recall from [PVS12, Lemma 2.7] that for a given spherical G -module W , the invariant d_W is easy to calculate from the rank of the free abelian group $\langle \mathcal{S}(W) \rangle_{\mathbb{Z}}$: it is the difference between the rank of $\langle \mathcal{S}(W) \rangle_{\mathbb{Z}}$ and the number of irreducible components of W . Thanks to the reduction in Section 4 of [PVS12], which is independent of the type of the group G , the proof of Theorem 1.1, which is formally given in Section 1.2, reduces to the following theorem.

Theorem 1.2. *Suppose (\overline{G}, W) is an entry in Knop's List of saturated indecomposable spherical modules (see List 3.1 on page 16). If G is a connected reductive group such that*

- (1) $\overline{G}' \subseteq G \subseteq \overline{G}$; and
- (2) W is spherical as a G -module

then

$$\dim T_{X_0} M_{\mathcal{S}}^G = d_W,$$

where \mathcal{S} is the weight monoid of (G, W) .

In [PVS12, Section 5], we proved Theorem 1.2 for groups \overline{G} of type A. In Section 3 below, we prove it for the remaining modules in Knop's List, i.e. those where the acting group contains a component that is not of type A. As in our previous paper, we do this by determining for each entry in Knop's List the structure of $T_{X_0} M_{\mathcal{S}}$ as a T_{ad} -module: we determine the T_{ad} -weights occurring in $T_{X_0} M_{\mathcal{S}}$ and show that each weight has multiplicity one. It follows from our descriptions that only certain 'special' elements of the root lattice of G occur as T_{ad} -weights in $T_{X_0} M_{\mathcal{S}}$: every T_{ad} -weight in $T_{X_0} M_{\mathcal{S}}$ is a so-called "spherical root" of G (cf. [Lun01, Section 1.2] for the definition of this notion).

Section 2, which may be of independent interest, contains some auxiliary results about the tangent space $T_{X_0} M_{\mathcal{S}}$ to $M_{\mathcal{S}}$ at the point X_0 . Corollary 2.9 is a sharpening of the extension criterion [PVS12, Proposition 3.4] for invariant sections of the normal sheaf of X_0 in V .

The Appendix presents the details, in a specific case, of a different technique which explicitly computes the T_{ad} -eigenvectors in $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$.

After most of the work on this paper had been completed, the preprints [ACF14] and [BVS15] were posted on the arXiv. We do not use the results contained in these papers, and for the weight monoids \mathcal{S} under consideration in the present paper our main result is stronger. More precisely, while [ACF14, BVS15] also prove (and in much greater generality than in the present paper) that the T_{ad} -weights in $T_{X_0}M_{\mathcal{S}}$ are spherical roots of G and have multiplicity one, in the present paper we additionally prove that $M_{\mathcal{S}}$ is irreducible for the monoids \mathcal{S} under consideration.

1.2. Formal proof of Theorem 1.1. We now give the proof of Theorem 1.1. Corollary 2.6 and Corollary 4.17 of [PVS12] reduce the proof to Theorem 1.2, which we prove by a case-by-case verification in Section 3.

1.3. Notations. We will follow the conventions and notations of [PVS12]. In particular, by a variety we mean a reduced, irreducible and separated scheme of finite type over \mathbb{C} . We will use Λ for the weight lattice of G , i.e. the group of characters of a fixed maximal torus T , which is identified with the group of characters of a chosen Borel subgroup B of G which contains T . Then Λ^+ will denote the monoid of dominant weights in Λ , and we will use $V(\lambda)$ for the irreducible representation of G corresponding to $\lambda \in \Lambda^+$, and v_λ for a highest weight vector in $V(\lambda)$. We will use \mathfrak{g} for the Lie algebra of G . If α is a root, then $\alpha^\vee \in \text{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ will be its coroot (in the sense of [Bou68]), \mathfrak{g}^α its root space and $X_\alpha \in \mathfrak{g}^\alpha \setminus \{0\}$ a root operator. We will use Π for the set of simple roots (relative to T and B) and Λ_R for the root lattice: $\Lambda_R = \langle \Pi \rangle_{\mathbb{Z}} \subseteq \Lambda$.

We will number the fundamental weights and the simple roots of the simple Lie algebras as in [Bou68]. When $G = \text{GL}(n)$ and $i \in \{1, \dots, n\}$, the highest weight of the module $\bigwedge^i \mathbb{C}^n$ will be denoted by ω_i . Moreover, we put $\omega_0 = 0$. It is well-known that the simple roots of $\text{GL}(n)$ have the following expressions in terms of the ω_i :

$$(1.1) \quad \alpha_i = -\omega_{i-1} + 2\omega_i - \omega_{i+1} \quad \text{for } i \in \{1, 2, \dots, n-1\}.$$

We will use E^* for the basis of a free monoid \mathcal{S} of dominant weights and $E := \{\lambda^* : \lambda \in E^*\}$. Here λ^* is the highest weight of the representation $V(\lambda)^*$ which is dual to $V(\lambda)$; that is: $V(\lambda^*) \simeq V(\lambda)^*$.

1.4. Acknowledgment. The authors thank Michel Brion for suggesting, during a 2011 visit of the second-named author to the University of Grenoble, the general strategy for the extension criterion in Section 2. They also thank an anonymous referee for pointing out a mistake in the proof of Proposition 3.12, for her/his very careful reading and for the numerous and detailed suggestions which improved the paper.

S. P. benefited from experiments with the computer algebra program Macaulay2 [GS]. For different parts of the project he was financially supported by the Portuguese Fundação para a Ciência e a Tecnologia through grant SFRH/BPD/22846/2005 of POCI2010/FEDER and by RIMS, Kyoto University, Japan.

B. V. S. received support from The City University of New York PSC-CUNY Research Award Program and from the National Science Foundation through grant number DMS-1407394. He also thanks Michel Brion and the Institut Fourier for their hospitality in the Summer of 2011.

2. CRITERION FOR EXTENSION OF SECTIONS

In this section, E^* is a set of linearly independent dominant weights of a complex connected reductive group G , and \mathcal{S} is the submonoid of Λ^+ generated by E^* . We do not assume that \mathcal{S} is the weight monoid of a spherical module. Like before, $E = \{\lambda : \lambda^* \in E^*\}$. As in [PVS12], we put

$$\begin{aligned} V &:= \oplus_{\lambda \in E} V(\lambda); \\ x_0 &:= \sum_{\lambda \in E} v_\lambda \in V; \\ X_0 &:= \overline{G \cdot x_0} \subseteq V \end{aligned}$$

and we denote by $\mathcal{N}_{X_0|V}$ the normal sheaf of X_0 in V .

Remark 2.1. We record some well-known facts about x_0 and X_0 that will be of use later in the paper.

- (a) Since E is linearly independent, $t \cdot x_0 = \langle v_\lambda : \lambda \in E \rangle_{\mathbb{C}}$.
- (b) X_0 is a spherical G -variety with weight monoid \mathcal{S} ; cf. [VP72, Theorem 6]
- (c) By [VP72, Theorem 8], the following map is a one-to-one correspondence between the set of subsets of E and the set of G -orbits in X_0 :

$$(D \subseteq E) \mapsto G \cdot v_D \quad \text{where } v_D := \sum_{\lambda \in D} v_\lambda.$$

In [AB05] Alexeev and Brion equipped $M_{\mathcal{S}}$ with an action of T_{ad} and showed that X_0 , viewed as a point of $M_{\mathcal{S}}$, is a fixed point and the unique closed orbit for this action. As in [PVS12] we will work with a ‘twist’ of the action in [AB05]. It is obtained by composing Alexeev and Brion’s action with the automorphism of T_{ad} induced by the automorphism $\gamma \mapsto w_0(\gamma)$ of the root lattice Λ_R , which is the group of characters of T_{ad} . Here w_0 is the longest element of the Weyl group of (G, T) . We will call our action on $M_{\mathcal{S}}$ and its induced action on $T_{X_0}M_{\mathcal{S}}$ “the T_{ad} -action.” As shown in [AB05] and reviewed in [PVS12, §2.2], we have a sequence of T_{ad} -equivariant linear maps

$$(2.1) \quad T_{X_0}M_{\mathcal{S}} \xrightarrow{\simeq} H^0(X_0, \mathcal{N}_{X_0|V})^G \hookrightarrow H^0(G \cdot x_0, \mathcal{N}_{X_0|V})^G \xrightarrow{\simeq} (V/\mathfrak{g} \cdot x_0)^{G_{x_0}},$$

where the first and the third map are isomorphisms, and the second one is an inclusion.

Because they will play a role later on, we recall from [PVS12, §2.2] explicit descriptions of our T_{ad} -actions on $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ and on $H^0(X_0, \mathcal{N}_{X_0|V})^G$. For the former, we begin by equipping V with the same action α of T_{ad} as in [PVS12, Definition 2.11]: if $t \in T, \lambda \in E$ and $v \in V(\lambda) \subseteq V$ then

$$(2.2) \quad \alpha(t, v) = \lambda(t)t^{-1}v.$$

It follows from highest weight theory that the center $Z(G)$ of G belongs to the kernel of α , and therefore α induces an action of $T_{\text{ad}} = T/Z(G)$ on V . Let $G \rtimes T_{\text{ad}}$ be the semidirect product of G and T_{ad} , where T_{ad} acts on G by conjugation. As explained in [AB05, p. 102] the T_{ad} -action α and the linear G -action on V can be extended together to a linear action of $G \rtimes T_{\text{ad}}$ on V . Since the T_{ad} -action fixes x_0 , this yields an action of $G_{x_0} \rtimes T_{\text{ad}}$ on $V/\mathfrak{g} \cdot x_0$, see e.g. [PVS12, p. 1780]. It follows that the subspace $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ of $V/\mathfrak{g} \cdot x_0$ is preserved

by the action of T_{ad} . This induced action on $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ is what we call “the T_{ad} -action” on $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$. By slight abuse of notation, we also denote it by α .

To describe the T_{ad} -action on $H^0(G \cdot x_0, \mathcal{N}_{X_0|V})^G$ and on $H^0(X_0, \mathcal{N}_{X_0|V})^G$, let $\text{GL}(V)^G$ be the group of linear automorphisms of V that commute with the action of G . Since the elements of E are distinct, $\text{GL}(V)^G$ is isomorphic to the product of $|E|$ copies of \mathbb{C}^\times . The natural action of $\text{GL}(V)^G$ on V stabilizes $G \cdot x_0$ and X_0 and the embedding $H^0(X_0, \mathcal{N}_{X_0|V})^G \hookrightarrow H^0(G \cdot x_0, \mathcal{N}_{X_0|V})^G$ is $\text{GL}(V)^G$ -equivariant for the induced actions. Composing the action of $\text{GL}(V)^G$ with the homomorphism

$$(2.3) \quad f: T \rightarrow \text{GL}(V)^G, t \mapsto (\lambda(t))_{\lambda \in E}$$

yields an action of T on V . We denote the induced T -action on $H^0(G \cdot x_0, \mathcal{N}_{X_0|V})^G$ and on $H^0(X_0, \mathcal{N}_{X_0|V})^G$ by $\hat{\psi}$. Proposition 2.13 of [PVS12] shows that $Z(G)$ is in the kernel of $\hat{\psi}$ and that the isomorphism

$$(2.4) \quad H^0(G \cdot x_0, \mathcal{N}_{X_0|V})^G \rightarrow (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}, s \mapsto s(x_0)$$

in (2.1) above is indeed T_{ad} -equivariant if $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ is equipped with the T_{ad} -action α and $H^0(G \cdot x_0, \mathcal{N}_{X_0|V})^G$ is equipped with the T_{ad} -action $\hat{\psi}$.

In Section 2.1 we strengthen [PVS12, Proposition 3.4] and obtain necessary and sufficient conditions for a section $s \in H^0(G \cdot x_0, \mathcal{N}_{X_0|V})^G$ to extend to X_0 : see Corollary 2.9. The proof is given in Section 2.3, after we review some generalities about extending sections of a vector bundle over a normal variety in Section 2.2. In Section 2.4 we gather a few more results on $T_{X_0}M_S$.

2.1. Extending sections. We denote by $X_0^{\leq 1} \subset X_0$ the union of $G \cdot x_0$ with all G -orbits of X_0 that have codimension 1. By [Bri10, Lemma 1.14] $X_0^{\leq 1}$ is an open subset of X_0 , and because X_0 is normal, it is a subset of the smooth locus of X_0 (see, e.g., the argument in the proof of [PVS12, Lemma 3.3] for details).

Definition 2.2. We say the $\lambda \in E$ has **codimension one** if

$$\dim G \cdot (x_0 - v_\lambda) = (\dim G \cdot x_0) - 1.$$

As an immediate consequence of, e.g., [PVS12, Proposition 3.1] one has the following simple criterion to determine whether an element of E has codimension one.

Proposition 2.3. *For $\lambda \in E$ the following are equivalent*

- (1) λ has codimension one;
- (2) for every $\alpha \in \Pi$ such that $\langle \alpha^\vee, \lambda \rangle \neq 0$, there exists $\mu \in E \setminus \{\lambda\}$ such that $\langle \alpha^\vee, \mu \rangle \neq 0$;
- (3) for every positive root α such that $\langle \alpha^\vee, \lambda \rangle \neq 0$, there exists $\mu \in E \setminus \{\lambda\}$ such that $\langle \alpha^\vee, \mu \rangle \neq 0$.

The following is an immediate consequence of [Bri13, Lemma 3.9].

Proposition 2.4. *If $s \in H^0(G \cdot x_0, \mathcal{N}_{X_0|V})^G$, then the following are equivalent:*

- (1) s extends to X_0 ;
- (2) s extends to $X_0^{\leq 1}$;
- (3) s extends to $G \cdot x_0 \cup G \cdot (x_0 - v_\lambda)$ for every $\lambda \in E$ of codimension 1.

Proof. The equivalence of (1) and (2) is a special case of [Bri13, Lemma 3.9]. The equivalence of (2) and (3) is a consequence of the definition of a sheaf, once we prove that the collection of sets

$$(2.5) \quad \{G \cdot x_0 \cup G \cdot (x_0 - v_\lambda) : \lambda \in E \text{ of codimension } 1\}$$

forms an open cover of $X_0^{\leq 1}$. We first show that each set $G \cdot x_0 \cup G \cdot (x_0 - v_\lambda)$ in the collection (2.5) is open. Indeed, the complement of $G \cdot x_0 \cup G \cdot (x_0 - v_\lambda)$ in $X_0^{\leq 1}$ is a finite (by Remark 2.1(c)) union of orbits in $X_0^{\leq 1}$ that are all closed because they are of minimal dimension. Secondly, the union of the sets in the collection (2.5) is all of $X_0^{\leq 1}$ because, by Remark 2.1(c) and [PVS12, Lemma 2.16] every orbit of codimension 1 in X_0 is of the form $G \cdot (x_0 - v_\lambda)$ for some $\lambda \in E$ of codimension 1. \square

We recall some well-known facts about T_{ad} -weights and T_{ad} -eigenvectors in $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$.

Lemma 2.5. *Let $\beta \in \Lambda$. If $v \in V$ such that $[v] \in (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ is a nonzero T_{ad} -eigenvector of T_{ad} -weight β , then there exists a T_{ad} -eigenvector $\hat{v} \in V$ of T_{ad} -weight β such that $[v] = [\hat{v}] \in (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$.*

Proof. This follows from the following standard argument. Note that $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ is a T_{ad} -stable subspace of $V/\mathfrak{g} \cdot x_0$. Moreover, since the subspace $\mathfrak{g} \cdot x_0$ of V is T_{ad} -stable, there exists another T_{ad} -stable subspace L of V such that $V = \mathfrak{g} \cdot x_0 \oplus L$. The restriction of the quotient map $V \rightarrow V/\mathfrak{g} \cdot x_0$ to L is an isomorphism $L \rightarrow V/\mathfrak{g} \cdot x_0$ of T_{ad} -modules. We can take \hat{v} to be the inverse image in L of $[v]$ under this isomorphism. \square

Proposition 2.6. *Let $\beta \in \Lambda$ and assume that v is a T_{ad} -eigenvector in V of weight β such that $0 \neq [v] \in (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$. Then*

- (a) *there exists $\alpha \in \Pi$ such that $X_\alpha \cdot v \neq 0$ and $\beta - \alpha \in R^+ \cup \{0\}$;*
- (b) *$\beta \in \langle E \rangle_{\mathbb{Z}}$;*
- (c) *$X_\alpha \cdot v \in \langle X_{-(\beta-\alpha)} \cdot x_0 \rangle_{\mathbb{C}}$ for all $\alpha \in \Pi$ such that $\beta - \alpha \in R^+$;*
- (d) *$X_\alpha \cdot v = 0$ for all $\alpha \in \Pi$ such that $\beta - \alpha \notin R^+ \cup \{0\}$.*

Proof. Assertion (b) is a consequence in Lemma 2.17(c) of [PVS12]. Let $\alpha \in \Pi$. Recalling that if $X_\alpha \cdot v$ is nonzero, then it has T_{ad} -weight $\beta - \alpha$, assertions (a) and (d) follow from Lemma 2.18 in *loc.cit.* Since the root operator X_α belongs to the Lie algebra of G_{x_0} , we have that $X_\alpha \cdot v \in \mathfrak{g} \cdot x_0$. Assertion (c) now follows from the fact that if the T_{ad} -weight $\beta - \alpha$ occurs in $\mathfrak{g} \cdot x_0$ then the corresponding weight space is $\langle X_{-(\beta-\alpha)} \cdot x_0 \rangle_{\mathbb{C}}$. \square

Remark 2.7. Since we will frequently make use of it later, we note the following consequence of (a) and (b) in Proposition 2.6: if β is a T_{ad} -weight in $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ then

$$(2.6) \quad 0 \neq \beta \in \langle \Pi \rangle_{\mathbb{N}} \cap \langle E \rangle_{\mathbb{Z}}.$$

Theorem 2.8. *Assume that v is a T_{ad} -eigenvector in V of T_{ad} -weight β such that*

$$0 \neq [v] \in (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}.$$

Denote by $s \in H^0(G \cdot x_0, \mathcal{N}_{X_0|V})^G$ the G -equivariant section defined by $s(x_0) = [v]$. Let λ be an element of E which has codimension 1 and put $Z = G \cdot x_0 \cup G \cdot (x_0 - v_\lambda)$. Let a be the coefficient of λ in the unique expression of β as a linear combination of elements of E .

- A) If $a \leq 0$, then s extends to an element of $H^0(Z, \mathcal{N}_{X_0|V})^G$.
 B) If $a > 1$, then s does not extend to an element of $H^0(Z, \mathcal{N}_{X_0|V})^G$.
 C) Assume $a = 1$. Then the following are equivalent:
 i) s extends to an element of $H^0(Z, \mathcal{N}_{X_0|V})^G$;
 ii) There exists $\hat{v} \in V(\lambda)$ such that $[v] = [\hat{v}]$ as elements of $V/\mathfrak{g} \cdot x_0$.

The proof of Theorem 2.8 will be given in Section 2.3 which starts on page 10. Before that, we gather some general results on extending sections in Section 2.2. The following is a synthesis of Proposition 2.4 and Theorem 2.8.

Corollary 2.9. Assume that v is a T_{ad} -eigenvector in V of T_{ad} -weight β such that

$$0 \neq [v] \in (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}.$$

Denote by $s \in H^0(G \cdot x_0, \mathcal{N}_{X_0|V})^G$ the G -equivariant section defined by $s(x_0) = [v]$. Let

$$(2.7) \quad \beta = \sum_{\lambda \in E} a_\lambda \lambda$$

be the unique expression of β as a \mathbb{Z} -linear combination of the elements of E .

The section s extends to an element of $H^0(X_0, \mathcal{N}_{X_0|V})^G$ if and only if for all $\lambda \in E$ of codimension 1 we have

- $a_\lambda \leq 1$; and
- if $a_\lambda = 1$ then there exists $\hat{v} \in V(\lambda)$ such that $[v] = [\hat{v}]$ as elements of $V/\mathfrak{g} \cdot x_0$.

Remark 2.10. It follows from Proposition 2.20 below that if s extends, then at most two of the a_λ in equation 2.7 are positive, irrespective of whether λ is of codimension 1 or not.

Example 2.11 (Luna). Let $G = \text{SL}_2 \times \text{SL}_2$, and $E = \{\lambda_1, \lambda_2\}$ with $\lambda_1 = 2\omega, \lambda_2 = 4\omega + 2\omega'$. Here ω is the fundamental weight of the first component of G , and ω' is that of the second component. Similarly, we will use α and α' for the simple root of the first and second component of G , respectively. Using Proposition 2.3, it follows that v_{λ_2} has a G -orbit of codimension 1 in X_0 , while v_{λ_1} has a G -orbit of codimension ≥ 2 . Hence by Proposition 2.4 for $[w] \in (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$, the equivariant section of $\mathcal{N}_{X_0|V}$ on $G \cdot x_0$ induced by $[w]$ extends to X_0 if and only if it extends over $G \cdot x_0 \cup G \cdot v_{\lambda_2}$.

Denote by e_1, e_2 (resp. g_1, g_2) a basis of \mathbb{C}^2 where the first (resp. second) SL_2 acts in the standard fashion. A small calculation gives that the vector space $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ is 3-dimensional with basis the classes in $V/\mathfrak{g} \cdot x_0$ of

$$w_1 = e_1 e_2, \quad w_2 = e_1^4 \otimes g_2^2, \quad w_3 = e_2^2 + e_1^2 e_2^2 \otimes g_1^2.$$

The vector w_1 has T_{ad} weight $\alpha = \lambda_1$, and since $w_1 \in V(\lambda_1)$ part C) of Theorem 2.8 implies that the induced equivariant section extends to $G \cdot x_0 \cup G \cdot v_{\lambda_2}$, hence to the whole of X_0 . The vector w_2 has T_{ad} weight $2\alpha' = -4\lambda_1 + 2\lambda_2$, and part A) of Theorem 2.8 implies that the induced equivariant section extends to X_0 . The vector w_3 has T_{ad} weight $2\alpha = 2\lambda_1$, hence part B) of Theorem 2.8 implies that the induced equivariant section does not extend to $G \cdot x_0 \cup G \cdot v_{\lambda_2}$. We have shown that

$$T_{X_0} \mathbf{M}_S = V(\alpha) \oplus V(2\alpha')$$

as a T_{ad} -module. We remark that to exclude the section induced by w_3 from $T_{X_0}M_S$ we could not have used [PVS12, Proposition 3.4], since condition (ES2) of that proposition is not satisfied for w_3 .

We also remark that Luna has shown in an unpublished note from 2005 that this moduli scheme M_S , equipped with its reduced scheme structure, is a union of two affine lines meeting in a point. It was the first example of a non-irreducible scheme M_S .

2.2. Generalities about extending sections. In this section X denotes a variety, in particular it is reduced, irreducible and separated. Let $\mathcal{E} \rightarrow X$ be an algebraic vector bundle. For the proof of Theorem 2.8 in Section 2.3 we need the following general propositions. They are well known, but for completeness we provide proofs.

Proposition 2.12. *Assume that X is normal, that $U \subseteq X$ is a nonempty Zariski open subset, and that $f : U \rightarrow \mathbb{C}$ a morphism. If f does not extend to a morphism $X \rightarrow \mathbb{C}$, then there exists $p \in X \setminus U$ such that for every irreducible algebraic curve $C \subseteq X$ with $p \in C$ and $U \cap C \neq \emptyset$ the morphism f restricted to $U \cap C$ does not extend to a morphism $C \rightarrow \mathbb{C}$.*

Proof. We consider f as a rational function on X . Since f does not extend to a morphism $X \rightarrow \mathbb{C}$ it follows that f is not the constant function with value 0. Using that X is normal, there is a well defined divisor of poles of f and a well defined divisor of zeroes of f , see, e.g. [CLS11, Section 4.0]. Since f does not extend to a morphism $X \rightarrow \mathbb{C}$ the divisor of poles of f is nonzero, see, e.g. [CLS11, Proposition 4.0.16].

We fix a point $p \in X$ which is in the support of the divisor of poles of f but not in the support of the divisor of zeroes. To obtain a contradiction, we assume that there exists an irreducible algebraic curve $C \subset X$ with $p \in C$ and $U \cap C \neq \emptyset$ such that the morphism f restricted to $U \cap C$ extends to a morphism $\bar{f} : C \rightarrow \mathbb{C}$.

Denote by g the rational function $1/f$. The divisor of zeroes of g is the divisor of poles of f and the divisor of poles of g is the divisor of zeroes of f . Hence p is in the support of the divisor of zeroes of g and is not in the support on the divisor of poles of g . It follows that there exists a Zariski open subset $W \subset X$ with $p \in W$ such that g defines a morphism $g : W \rightarrow \mathbb{C}$ with the property $g(p) = 0$. Denote by $\bar{g} : W \cap C \rightarrow \mathbb{C}$ the restriction of g to $W \cap C$. We have that $W \cap U$ is a nonempty Zariski open subset of X . Since $W \cap C \neq \emptyset$ and $U \cap C \neq \emptyset$ we get that $W \cap U \cap C \neq \emptyset$. For $q \in W \cap U \cap C$ we have $(\bar{f}\bar{g})(q) = (fg)(q) = 1$. Since $W \cap U \cap C$ is dense in C it follows that $(\bar{f}\bar{g})(p) = 1$ which contradicts $\bar{g}(p) = 0$. \square

Example 2.13. If $X = \mathbb{C}^2$ and $f = x/y$ we can choose $p = (a, 0)$ for any $a \in \mathbb{C} \setminus \{0\}$.

Proposition 2.14. *Assume that X is normal, that $U \subset X$ is a nonempty Zariski open subset, and that $s \in H^0(U, \mathcal{E})$ is a section of the vector bundle \mathcal{E} . If s does not extend to a section $X \rightarrow \mathcal{E}$, then there exists $p \in X \setminus U$ such that for every algebraic curve $C \subset X$ with $p \in C$ and $U \cap C \neq \emptyset$ the section s restricted to $U \cap C$ does not extend to a section $C \rightarrow \mathcal{E}$.*

Proof. By the defining gluing property of sections of sheaves, there exists a nonempty Zariski open $V \subset X$ such that \mathcal{E} restricted to V is trivial and s restricted to $V \cap U$ does not extend to a section over V . Hence we can assume, without loss of generality, that \mathcal{E} is the trivial vector bundle. So assume $\mathcal{E} = X \times \mathbb{C}^n \rightarrow X$ is the first projection. Let e_1, \dots, e_n be a basis of \mathbb{C}^n and define $s_i \in H^0(X, \mathcal{E})$ by $s_i(x) = (x, e_i)$ for all $x \in X$. There exist (unique)

morphisms $f_i : U \rightarrow \mathbb{C}$ such that

$$s(u) = \sum_{i=1}^n f_i(u) s_i(u)$$

for all $u \in U$. If each f_i extended to a morphism $X \rightarrow \mathbb{C}$, it would then follow that s extends to a section $X \rightarrow \mathcal{E}$ which contradicts the assumptions. Hence at least one of the f_i does not extend to a morphism. Using Proposition 2.12 there exists $p \in X \setminus U$ such that for every algebraic curve $C \subset X$ with $p \in C$ and $U \cap C \neq \emptyset$, the morphism f_i restricted to $U \cap C$ does not extend to a morphism $C \rightarrow \mathbb{C}$. As a consequence, s restricted to $U \cap C$ does not extend to a section $C \rightarrow \mathcal{E}$. \square

Assume now in addition that G is a connected linear algebraic group over \mathbb{C} , X is a G -variety and $\pi : \mathcal{E} \rightarrow X$ is a G -vector bundle over X . This means that we are given an algebraic action $\rho : G \times \mathcal{E} \rightarrow \mathcal{E}$ such that

$$\rho(g, v) \in \pi^{-1}(g \cdot (\pi(v)))$$

for all $g \in G, v \in V$ and that for fixed $g \in G$ and $x \in X$ the induced map

$$\pi^{-1}(x) \rightarrow \pi^{-1}(g \cdot x), \quad v \mapsto \rho(g, v)$$

is an isomorphism of vector spaces.

While its proof is elementary, the following proposition implies that the section s of Theorem 2.8 extends to Z if and only if it extends along just one curve; see Proposition 2.16.

Proposition 2.15. *Assume that X is normal, that $U \subset X$ is a nonempty G -stable Zariski open subset such that $X \setminus U$ is a single G -orbit, and that $s \in H^0(U, \mathcal{E})^G$ is a G -equivariant section $U \rightarrow \mathcal{E}$. Assume that there exists $p_0 \in X \setminus U$ and an algebraic curve $C_0 \subset X$ with $p_0 \in C_0$ and $U \cap C_0 \neq \emptyset$ such that s restricted to $U \cap C_0$ extends to a section $s_0 : C_0 \rightarrow \mathcal{E}$. Then s extends to an element of $H^0(X, \mathcal{E})^G$.*

Proof. We first show that s extends to an element of $H^0(X, \mathcal{E})$, and then that the extension is G -equivariant.

We assume that s does not extend to an element of $H^0(X, \mathcal{E})$, and we will get a contradiction. By Proposition 2.14 there exists $p \in X \setminus U$ such that for every algebraic curve $C \subset X$ with $p \in C$ and $U \cap C \neq \emptyset$ the section s restricted to $U \cap C$ does not extend to a section $C \rightarrow \mathcal{E}$. Since $X \setminus U$ is a single G -orbit, there exists $g \in G$ with $g \cdot p_0 = p$.

Set $C = \{g \cdot v : v \in C_0\}$ and define $t : C \rightarrow \mathcal{E}$ by

$$t(v) = g \cdot s_0(g^{-1} \cdot v)$$

for $v \in C$. Since s is G -equivariant we have $t|_{U \cap C} = s|_{U \cap C}$, hence t is a section of \mathcal{E} over C which extends $s|_{U \cap C}$, contradicting the choice of p .

We have shown that s extends to a section $s_1 : X \rightarrow \mathcal{E}$. We claim that s_1 is G -equivariant. Indeed, define $s_2 : X \rightarrow \mathcal{E}$ by $s_2(v) = g \cdot s_1(g^{-1} \cdot v)$. Since s is G -equivariant on U , we have that $s_2(u) = s(u) = s_1(u)$ for all $u \in U$. As a consequence $s_2 = s_1$, which implies that s_1 is G -equivariant. \square

2.3. Proof of Theorem 2.8. We start the proof of Theorem 2.8. Let $\lambda \in E$ be of codimension 1. For $t \in \mathbb{C}$, we put

$$(2.8) \quad z_t := t \cdot v_\lambda + \sum_{\mu \in E \setminus \{\lambda\}} v_\mu.$$

Note that, because E is linearly independent, $z_t \in T \cdot x_0 \subseteq G \cdot x_0$ for $t \in \mathbb{C} \setminus \{0\}$. Moreover,

$$(2.9) \quad z_0 = \sum_{\mu \in E \setminus \{\lambda\}} v_\mu = x_0 - v_\lambda;$$

and $G \cdot z_0$ has codimension 1 in X_0 .

Since $Z = G \cdot x_0 \cup G \cdot z_0$ is smooth, the restriction of the sheaf $\mathcal{N}_{X_0|V}$ to Z is locally free. We denote by $\mathcal{E} \rightarrow Z$ the total space of the corresponding vector bundle. In particular the sections of the restriction of $\mathcal{N}_{X_0|V}$ to Z are naturally identified with those of \mathcal{E} .

Recall that $s \in H^0(G \cdot x_0, \mathcal{E})^G$ denotes the equivariant section induced by v ; that is $s(x_0) = [v] \in (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$. Set $C_0 = \{z_t : t \in \mathbb{C}\}$, and denote by s^* the section of \mathcal{E} over $G \cdot x_0 \cap C_0$ defined by $s^*(z_t) = s(z_t)$.

Proposition 2.16. *The section s extends to an element of $H^0(Z, \mathcal{E})^G$ if and only if s^* extends to a section of \mathcal{E} over C_0 .*

Proof. If s^* extends, then so does s by Proposition 2.15. The converse is obvious. \square

For $w \in V$ we denote by $s_w \in H^0(X_0, \mathcal{N}_{X_0|V})$ the global section defined by

$$s_w(x) = [w] \in V/T_x X_0$$

for all $x \in X_0$. We will use V^β for the T_{ad} -weight space in V of weight β . Recall that, by Proposition 2.6, β is a nonzero element of $\langle \Pi \rangle_{\mathbb{N}}$ and that $v \in V^\beta$ such that $0 \neq [v] \in (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$.

The idea of the proof of Theorem 2.8 is to find elements $\{y_i\}$ in V^β such that their images in $V/T_{z_t} X_0$ form a basis of the T_{ad} -weight space of weight β in $V/T_{z_t} X_0$ for all $t \in \mathbb{C}$. It then follows that (the restriction to C_0 of) the sections s_{y_i} form a linearly independent subset of $H^0(C_0, \mathcal{E})$, and that there exist $f_i \in \mathbb{C}(t)$ such that for all $t \in \mathbb{C} \setminus \{0\}$ we have

$$(2.10) \quad s^*(z_t) = \sum_i f_i(t) s_{y_i}(z_t).$$

The section s^* extends to all of C_0 if and only if each $f_i(t)$ belongs to the polynomial ring $\mathbb{C}[t]$. With the appropriate choice of the vectors $\{y_i\}$ the rational functions $f_i(t)$ are very simple; see Proposition 2.18.

By [PVS12, Lemma 3.3] $T_{z_0} X_0$ is the linear span of $\mathfrak{g} \cdot z_0 \cup \{v_\lambda\}$. If t is nonzero, then $G \cdot z_t$ is open in X_0 , whence $T_{z_t} X_0 = \mathfrak{g} \cdot z_t$, and $v_\lambda \in T_{z_t} X_0$, by Remark 2.1(a). The image of V^β under the projection $V \rightarrow V/T_{z_t} X_0$ can naturally be identified with $V^\beta / (V^\beta \cap T_{z_t} X_0)$.

Lemma 2.17. *Assume $t \in \mathbb{C}$. If β is not a root, then $V^\beta \cap T_{z_t} X_0 = 0$. If β is a root, then $V^\beta \cap T_{z_t} X_0$ is equal to $\langle X_{-\beta} \cdot z_t \rangle_{\mathbb{C}}$, so it is either 0 or 1-dimensional.*

Proof. Recall that $T_{z_t} X_0$ is equal to the linear span of $\mathfrak{g} \cdot z_t \cup \{v_\lambda\}$. Using that \mathfrak{u}^- is spanned by the set $\{X_{-\gamma} : \gamma \text{ positive root of } G\}$, the lemma follows from the fact that v_λ has T_{ad} -weight zero (and so not equal to β) and that $\mathfrak{g} \cdot z_t = \mathfrak{t} \cdot z_t \oplus \mathfrak{u}^- \cdot z_t$. \square

Before giving the details of the remaining arguments for the proof of Theorem 2.8, we introduce some more notation for the remainder of this section. Put

$$V_1 := V(\lambda); \quad V_2 := \bigoplus_{\mu \in E \setminus \{\lambda\}} V(\mu).$$

Note that $V = V_1 \oplus V_2$. Set $n = \dim V^\beta + 1$ and $m = n - \dim V^\beta \cap V_1$.

Because the summands V_1 and V_2 of V are stable under the T_{ad} -action, there exists a basis y_2, \dots, y_n of V^β , such that $y_i \in V^\beta \cap V_2$ for $2 \leq i \leq m$ and $y_i \in V^\beta \cap V_1$ for $m+1 \leq i \leq n$. Since $v \in V^\beta$, there exist $b_i \in \mathbb{C}$ such that

$$(2.11) \quad v = \sum_{i=2}^n b_i y_i.$$

Recall that a is the coefficient of λ in the unique expression of β as a linear combination of the elements of E .

Proposition 2.18. *Let $\{y_2, y_3, \dots, y_n\}$ be a basis of V^β as above, and let b_i be elements of \mathbb{C} such that the equality (2.11) holds. For all $t \in \mathbb{C}^*$ we have*

$$s(z_t) = t^{-a} \left(\sum_{i=2}^m b_i s_{y_i}(z_t) \right) + t^{-a+1} \left(\sum_{i=m+1}^n b_i s_{y_i}(z_t) \right).$$

Proof. By assumption, s is an eigenvector of weight β for the T_{ad} -action $\hat{\psi}$ on $H^0(G \cdot x_0, \mathcal{E})^G \cong (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$. Equivalently, s is an eigenvector (of weight described below) for the natural action of $\text{GL}(V)^G$ on $H^0(G \cdot x_0, \mathcal{E})^G$ described in [PVS12, p. 1777] and recalled at the start of Section 2 above.

Set $D = E \setminus \{\lambda\}$ and recall the map $\sigma_D : \mathbb{C}^\times \rightarrow \text{GL}(V)^G$ of [PVS12, p. 1784]: for $t \in \mathbb{C}^\times$, the element $\sigma_D(t)$ of $\text{GL}(V)^G$ is defined by $\sigma_D(t) \cdot (w_1 + w_2) = tw_1 + w_2$, for all $w_1 \in V_1$ and $w_2 \in V_2$.

We now argue as in the proof of Part i) of [PVS12, Proposition 3.4]. The homomorphism $f : T \rightarrow \text{GL}(V)^G$ of (2.3) is surjective (because E is linearly independent), and therefore the homomorphism $f^* : X(\text{GL}(V)^G) \rightarrow X(T)$, $\delta \mapsto \delta \circ f$ of character groups is injective. Then $\delta := (f^*)^{-1}(\beta)$ is the $\text{GL}(V)^G$ -weight of s . Consequently

$$\begin{aligned} s(z_t) &= s(\sigma_D(t) \cdot x_0) = \delta(\sigma_D(t))^{-1} \sigma_D(t) \cdot s(x_0) \\ &= \delta(\sigma_D(t))^{-1} [\sigma_D(t) \cdot v] \in V/\mathfrak{g} \cdot z_t. \end{aligned}$$

Since, by definition, a is the coefficient of λ in the expression of β as a \mathbb{Z} -linear combination of the elements of E , we have that $\delta(\sigma_D(t)) = t^a$. Consequently

$$s(z_t) = t^{-a} [\sigma_D(t) \cdot v].$$

Taking into account that

$$\sigma_D(t) \cdot v = \sum_{i=2}^m b_i y_i + t \sum_{i=m+1}^n b_i y_i.$$

the proposition follows. □

We prove part A) of Theorem 2.8. Assume $a \leq 0$. Since the s_{y_i} are sections defined over the whole X_0 , Proposition 2.18 implies that s^* extends to a section over C_0 . Hence s extends to Z by Proposition 2.16.

For the rest of the proof we separate into five cases. Recall that $x_0 = \sum_{\lambda \in E} v_\lambda$.

Case 1: β is a root, $X_{-\beta} \cdot v_\lambda \neq 0$ and there exists $\mu \in E \setminus \{\lambda\}$ with $X_{-\beta} \cdot v_\mu \neq 0$.

Case 2: β is a root, $X_{-\beta} \cdot v_\lambda = 0$ and there exists $\mu \in E \setminus \{\lambda\}$ with $X_{-\beta} \cdot v_\mu \neq 0$.

Case 3: β is a root, $X_{-\beta} \cdot v_\lambda \neq 0$ and $X_{-\beta} \cdot v_\mu = 0$ for all $\mu \in E \setminus \{\lambda\}$.

Case 4: β is a root, $X_{-\beta} \cdot v_\lambda = 0$ and $X_{-\beta} \cdot v_\mu = 0$ for all $\mu \in E \setminus \{\lambda\}$.

Case 5: β is not a root.

We first show that Case 3 and Case 4 cannot happen. Case 3 cannot occur, because it contradicts the assumption that λ has codimension 1, by Proposition 2.3. Case 4 cannot occur, because $\beta \in \langle E \rangle_{\mathbb{Z}}$ by Proposition 2.6. Indeed, if β is a root in $\langle E \rangle_{\mathbb{Z}}$, then $\{\mu \in E : \langle \beta^\vee, \mu \rangle \neq 0\}$ is nonempty and so $X_{-\beta} \cdot x_0 \neq 0$.

We now prove B) and C) of Theorem 2.8 in Case 1. We begin by choosing an appropriate basis of V^β . Put $y_n := X_{-\beta} \cdot v_\lambda$, and let y_2, y_3, \dots, y_q be the elements of

$$\{X_{-\beta} \cdot v_\mu : \mu \in E \setminus \{\lambda\}, \langle \beta^\vee, \mu \rangle \neq 0\}$$

in some order. Finally, extend $y_2, y_3, \dots, y_q, y_n$ to a basis y_2, y_3, \dots, y_n of V^β such that $y_i \in V_2$ when $q+1 \leq i \leq m$ and $y_i \in V_1$ when $m+1 \leq i \leq n-1$.

Assume $t \in \mathbb{C}$. Since $\mathfrak{g} \cdot z_t \subset T_{z_t} X_0$, we have $X_{-\beta} \cdot z_t \in T_{z_t} X_0$. Hence there is, in $V/T_{z_t} X_0$, the following equality

$$(2.12) \quad [y_2] = - \sum_{i=3}^q [y_i] - t[y_n].$$

Using Lemma 2.17 it follows that the classes, in $V/T_{z_t} X_0$, of y_3, \dots, y_n are a basis for the image of V^β in $V/T_{z_t} X_0$. In other words, the elements $s_{y_i}(z_t)$, for $3 \leq i \leq n$, are linearly independent for every $t \in \mathbb{C}$.

Combining the relation (2.12) with Proposition 2.18 we get for all nonzero t that

$$(2.13) \quad s(z_t) = t^{-a} \left(\sum_{i=3}^q (b_i - b_2) s_{y_i}(z_t) \right) + t^{-a} \left(\sum_{i=q+1}^m b_i s_{y_i}(z_t) \right) + \\ t^{-a+1} \left(\sum_{i=m+1}^{n-1} b_i s_{y_i}(z_t) \right) + t^{-a+1} (b_n - b_2) s_{y_n}(z_t).$$

We now prove part B) in Case 1. Assume $a > 1$. We assume that s extends and we will get a contradiction. Since s extends we have that s^* also extends. Since the set $\{s_{y_i}(z_t) : 3 \leq i \leq n\}$ is linearly independent for all $t \in \mathbb{C}$, and $-a+1$ and $-a$ are negative, Equation (2.13) implies that $b_i = b_2$ for $3 \leq i \leq q$ and for $i = n$, and that $b_i = 0$ for $q+1 \leq i \leq n-1$. Hence $v = b_2(X_{-\beta} \cdot x_0)$, contradicting the assumption $v \notin \mathfrak{g} \cdot x_0$. This proves part B) in Case 1.

We now prove part C) in Case 1. Assume that $a = 1$. Since $-a+1 = 0$ and $-a < 0$, arguing similarly to the case $a > 1$ we get that s^* extends over C_0 if and only if $b_i = b_2$ for

$3 \leq i \leq q$ and $b_i = 0$ for $q+1 \leq i \leq m$. If these conditions hold then $v - \hat{v} \in \mathfrak{g} \cdot x_0$, where

$$\hat{v} = \sum_{i=m+1}^{n-1} b_i y_i + (b_n - b_2) y_n,$$

which is an element of V_1 . Conversely, assume there exists $\hat{v} \in V_1$ with $v - \hat{v} \in \mathfrak{g} \cdot x_0$. Then v and \hat{v} define the same equivariant section s of \mathcal{E} over $G \cdot x_0$. Hence we can assume in Equation (2.11) that $b_i = 0$ for $2 \leq i \leq m$. As a consequence, Proposition 2.18 implies that s^* extends, hence by Proposition 2.16 s also extends. This finishes the proof of part C) of Theorem 2.8 in Case 1.

The arguments for the proof of B) and C) of Theorem 2.8 for Cases 2 and 5 are very similar to those of Case 1 so we only sketch them.

In Case 2, we can choose a basis y_2, y_3, \dots, y_n of V^β with the following properties:

- $\{y_2, y_3, \dots, y_q\} = \{X_{-\beta} \cdot v_\mu : \mu \in E, \langle \beta^\vee, \mu \rangle \neq 0\}$;
- $y_i \in V_2$ for $q+1 \leq i \leq m$; and
- $y_i \in V_1$ for $m+1 \leq i \leq n$.

Then, for all $t \in \mathbb{C}$ we have the following equality in $V/T_{z_t}(X_0)$:

$$[y_2] = - \sum_{i=3}^q [y_i]$$

and, similarly to Case 1, the elements $s_{y_i}(z_t)$, for $3 \leq i \leq n$, are linearly independent. Continuing as in the proof of Case 1 the result follows.

In Case 5, we choose a basis y_2, \dots, y_n of V^β with the following properties:

- $y_i \in V_2$ for $2 \leq i \leq m$; and
- $y_i \in V_1$ for $m+1 \leq i \leq n$.

Then for all $t \in \mathbb{C}$ the elements $s_{y_i}(z_t)$, for $2 \leq i \leq n$, are linearly independent. Continuing as in the proof of Case 1 the result follows. This finishes the proof of Theorem 2.8.

2.4. A few more facts about $T_{X_0}M_S$. In this subsection we prove three more facts about T_{ad} -weights in $T_{X_0}M_S$. Let β be such a weight, and let $\beta = \sum_{\lambda \in E} a_\lambda \lambda$ be the unique expression of β as a \mathbb{Z} -linear combination of the elements of E . Proposition 2.19 guarantees that at least one of the a_λ is positive. Proposition 2.20, which is a consequence of a classical result attributed to Kostant, bounds the number and size of positive coefficients a_λ . Finally, Proposition 2.21 gives a sufficient condition for β to be a simple root and describes the T_{ad} -weight space of weight β when the condition is met. The first two propositions do not use our extension criterion (Theorem 2.8), while the third one does.

Proposition 2.19. *Let β be a T_{ad} -weight in $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$. Then there exists a simple root α such that $\langle \alpha^\vee, \beta \rangle > 0$. Consequently, if $\beta = \sum_{\lambda \in E} a_\lambda \lambda$ is the unique expression of β as a \mathbb{Z} -linear combination of the elements of E , then there exists $\lambda \in E$ with $\langle \alpha^\vee, \lambda \rangle > 0$ and $a_\lambda > 0$.*

Proof. This follows by a standard argument from the fact, recalled in Proposition 2.6, that β is a nonzero element of $\langle \Pi \rangle_{\mathbb{N}}$. For completeness, we include the details. Recall that we can equip the vector space $\Lambda_R \otimes_{\mathbb{Z}} \mathbb{R}$, where $\Lambda_R = \langle \Pi \rangle_{\mathbb{Z}}$ is the root lattice, with a positive definite inner product $(\cdot | \cdot)$ such that for all $\alpha \in \Pi$ and all $\gamma \in \Lambda_R$, we have that $\langle \alpha^\vee, \gamma \rangle$ is a positive multiple of $(\alpha | \gamma)$, see e.g. [TY05, §18.3 and §18.4].

Since β is an element of $\langle \Pi \rangle_{\mathbb{N}}$ there exists, for every $\alpha \in \Pi$, a nonnegative integer n_α such that $\beta = \sum_{\alpha \in \Pi} n_\alpha \alpha$. Since $\beta \neq 0$, we know by the positive definiteness and the bilinearity of $(\cdot | \cdot)$ that

$$(2.14) \quad 0 < (\beta | \beta) = \sum_{\alpha \in \Pi} n_\alpha (\alpha | \beta).$$

It follows that there is some $\alpha \in \Pi$ for which $(\alpha | \beta) > 0$, whence $\langle \alpha^\vee, \beta \rangle > 0$, which is what we had to prove. \square

Proposition 2.20. *Let β be a T_{ad} -weight in $T_{X_0}M_S$. If $\beta = \sum_{\lambda \in E} a_\lambda \lambda$ is the unique expression of β as a \mathbb{Z} -linear combination of the elements of E , then the sum of the elements of the set*

$$\{a_\lambda : \lambda \in E \text{ and } a_\lambda > 0\}$$

is at most 2.

Proof. This is a consequence of the following fact, attributed to Kostant (see, e.g., [Tim11, Proposition 28.6]): the ideal $I = I(X_0)$ of the subvariety X_0 of V is generated by the intersection, which we denote by I_2 , of I with the subspace $\mathbb{C}[V]_2$ of polynomials of degree 2 in $\mathbb{C}[V]$. If we number the elements of E as $\{\lambda_1, \lambda_2, \dots, \lambda_p\}$, then

$$(2.15) \quad \mathbb{C}[V]_2 \cong [\oplus_{i=1}^p S^2 V(\lambda_i)] \oplus [\oplus_{1 \leq i < j \leq p} V(\lambda_i) \otimes V(\lambda_j)]$$

as G -modules.

It is shown in [AB05] (and reviewed in [PVS12, Section 2.1]) that because E is linearly independent M_S can be identified with an open subscheme of the invariant Hilbert scheme $\text{Hilb}_S^G(V)$. It therefore follows from [AB05, Proposition 1.13] (and its proof) that we have natural isomorphisms

$$(2.16) \quad T_{X_0}M_S \cong \text{Hom}_{\mathbb{C}[X_0]}^G(I/I^2, \mathbb{C}[X_0]) \cong \text{Hom}_{\mathbb{C}[V]}^G(I, \mathbb{C}[V]/I).$$

Recall from [PVS12, Section 2.2] that, as reviewed at the start of Section 2 above, the T_{ad} -action on $T_{X_0}M_S \cong \text{Hom}_{\mathbb{C}[V]}^G(I, \mathbb{C}[V]/I)$ is induced by the action of $\text{GL}(V)^G$ on V , using the homomorphism $f : T \rightarrow \text{GL}(V)^G$ of (2.3) on page 5. The $\text{GL}(V)^G$ -action on $\text{Hom}_{\mathbb{C}[V]}^G(I, \mathbb{C}[V]/I)$ induced by the $\text{GL}(V)^G$ -action on V is given by

$$(t \cdot \rho)(h) = t \cdot \rho(t^{-1} \cdot h)$$

for $\rho \in \text{Hom}_{\mathbb{C}[V]}^G(I, \mathbb{C}[V]/I)$, $t \in \text{GL}(V)^G$ and $h \in I$. Clearly, ρ is completely determined by its restriction to I_2 , since I_2 generates I as an ideal. Being G -equivariant, ρ sends each irreducible G -submodule M of I_2 to 0 or to a G -submodule of $\mathbb{C}[V]/I$ isomorphic to M .

Since $\mathbb{C}[V]/I = \mathbb{C}[X_0]$, the G -module structure of this algebra is given by

$$\mathbb{C}[V]/I \cong \oplus_{\mu \in S} V(\mu) = \oplus_{(b_i) \in \mathbb{N}^p} V(\sum_{i=1}^p b_i \lambda_i^*).$$

Moreover, as stated in the first paragraph of the proof of [Tim11, Proposition 28.6], $V(\sum_i b_i \lambda_i^*) \subseteq \mathbb{C}[V]/I$ is the image of $S^{b_1} V(\lambda_1^*) \otimes S^{b_2} V(\lambda_2^*) \otimes \dots \otimes S^{b_p} V(\lambda_p^*) \subseteq \mathbb{C}[V]$ under the quotient map $\mathbb{C}[V] \twoheadrightarrow \mathbb{C}[V]/I$. It follows that $t = (t_1, t_2, \dots, t_p) \in \text{GL}(V)^G$ acts on $x \in$

$V(\sum_i b_i \lambda_i^*) \subseteq \mathbb{C}[V]/I$ by

$$(2.17) \quad t \cdot x = t_1^{-b_1} t_2^{-b_2} \dots t_p^{-b_p} x,$$

since $t \mapsto t_1^{-b_1} t_2^{-b_2} \dots t_p^{-b_p}$ is the character by which $\mathrm{GL}(V)^G$ acts on $S^{b_1} V(\lambda_1^*) \otimes S^{b_2} V(\lambda_2^*) \otimes \dots \otimes S^{b_p} V(\lambda_p^*)$.

Now, suppose that ρ is a T_{ad} -eigenvector in $\mathrm{Hom}_{\mathbb{C}[V]}^G(I, \mathbb{C}[V]/I)$ of weight β , and let $\beta = \sum_{i=1}^p a_i \lambda_i$ be the expression of β as a \mathbb{Z} -linear combination of the elements of E . The homomorphism $f : T \rightarrow \mathrm{GL}(V)^G$ in (2.3) on page 5 relates the T_{ad} -action to the action of $\mathrm{GL}(V)^G$. Since $\beta \in \langle E \rangle_{\mathbb{Z}}$ and E is linearly independent there exists a unique character δ of $\mathrm{GL}(V)^G$ such that $\delta \circ f = \beta$. If $t = (t_1, \dots, t_p) \in \mathrm{GL}(V)^G$, then $\delta(t) = t_1^{a_1} t_2^{a_2} \dots t_p^{a_p}$. Moreover, δ is the $\mathrm{GL}(V)^G$ -weight of ρ , which means that for $t = (t_1, t_2, \dots, t_p) \in \mathrm{GL}(V)^G$, we have

$$(2.18) \quad t \cdot \rho = t_1^{a_1} t_2^{a_2} \dots t_p^{a_p} \rho.$$

Since $\rho \neq 0$ there exists an irreducible submodule M of I_2 and an element h of M such that $\rho(h) \neq 0$. Because $I_2 \subseteq \mathbb{C}[V]_2$, it follows from the decomposition (2.15) that there exist $i, j \in \{1, 2, \dots, p\}$, not necessarily distinct, such that for $t = (t_1, t_2, \dots, t_p) \in \mathrm{GL}(V)^G$ we have

$$(2.19) \quad t \cdot h = t_i^{-1} t_j^{-1} h,$$

since this is the action of $\mathrm{GL}(V)^G$ on $\mathbb{C}[V]_2$. Since $\rho(M) \neq 0$, there exists $(b_i) \in \mathbb{N}^p$ such that $\rho(M) = V(\sum_{i=1}^p b_i \lambda_i^*) \subseteq \mathbb{C}[V]/I$.

We then have for all $t = (t_1, \dots, t_p) \in \mathrm{GL}(V)^G$ that

$$(2.20) \quad (t \cdot \rho)(h) = t \cdot \rho(t^{-1} \cdot h)$$

$$(2.21) \quad = t_1^{-b_1} t_2^{-b_2} \dots t_p^{-b_p} (\rho(t_i t_j h))$$

$$(2.22) \quad = t_i t_j t_1^{-b_1} t_2^{-b_2} \dots t_p^{-b_p} (\rho(h))$$

where the second equality uses equations (2.17) and (2.19) and the third equality uses the \mathbb{C} -linearity of ρ . The proposition now follows from comparing (2.22) with equation (2.18). \square

Proposition 2.21. *Let β be a T_{ad} -weight in $T_{X_0} \mathcal{M}_{\mathcal{S}}$. If there exists $\lambda \in E$ satisfying the following two properties:*

- λ is of codimension one;
- λ has a positive coefficient in the unique expression of β as a linear combination of the elements of E ;

then β is a simple root. Moreover $\langle \beta^\vee, \lambda \rangle \neq 0$ and the T_{ad} -weight space in $H^0(X_0, \mathcal{N}_{X_0|V})^G \simeq T_{X_0} \mathcal{M}_{\mathcal{S}}^G$ of weight β is spanned by the section induced by $[X_{-\beta} v_\lambda] \in (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$.

Proof. Since β is a T_{ad} -weight in $T_{X_0} \mathcal{M}_{\mathcal{S}} \simeq H^0(X_0, \mathcal{N}_{X_0|V})^G$, there exists $v \in V$ of T_{ad} -weight β such that $[v]$ is a nonzero element of $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ and such that the corresponding section in $H^0(G \cdot x_0, \mathcal{N}_{X_0|V})^G$ extends to $Z = G \cdot x_0 \cup G \cdot (x_0 - v_\lambda)$. By Theorem 2.8(C),

we may assume that $v \in V(\lambda) \subseteq V$. To get a contradiction, we assume that β is not a simple root. Then, by Proposition 2.6, there exists a simple root α such that $\beta - \alpha$ is a positive root and

$$(2.23) \quad 0 \neq X_\alpha \cdot v \in \langle X_{-(\beta-\alpha)} x_0 \rangle_{\mathbb{C}}.$$

Because $v \in V(\lambda)$, we have that

$$(2.24) \quad X_\alpha \cdot v \in V(\lambda).$$

On the other hand, because λ is of codimension one, there exists $\lambda' \in E \setminus \{\lambda\}$ such that $\langle (\beta - \alpha)^\vee, \lambda' \rangle \neq 0$. Consequently, the line $\langle X_{-(\beta-\alpha)} x_0 \rangle_{\mathbb{C}}$ has nonzero projection on $V(\lambda')$. We have shown that (2.23) and (2.24) are in contradiction. That is, we have shown that β is a simple root.

By elementary highest weight theory, the T_{ad} -weight space of weight β in $V(\lambda)$ is $\langle X_{-\beta} v_\lambda \rangle_{\mathbb{C}}$. This implies the second assertion. \square

3. PROOF OF THEOREM 1.2

In this section, we prove Theorem 1.2 through case-by-case verification: we verify that the theorem holds for each saturated indecomposable spherical module in List 3.1 below. The definition of ‘saturated’ and ‘indecomposable’ can be found in [Kno98, Section 5] or in [PVS12, Definition 4.1]. The eight families (K1), (K2), (K3), (K15), (K16), (K17), (K18) and (K21) were the subject of [PVS12, Section 5]. Each subsection of this section corresponds to one of the remaining families in the list: the proposition in each subsection asserts that Theorem 1.2 holds for the family under consideration.

List 3.1 (Knop’s List [Kno98, Section 5]). The saturated indecomposable spherical modules (\overline{G}, W) are

- (K1) $(\text{GL}(m) \times \text{GL}(n), \mathbb{C}^m \otimes \mathbb{C}^n)$ with $1 \leq m \leq n$;
- (K2) $(\text{GL}(n), \text{Sym}^2 \mathbb{C}^n)$ with $1 \leq n$;
- (K3) $(\text{GL}(n), \wedge^2 \mathbb{C}^n)$ with $2 \leq n$;
- (K4) $(\text{Sp}(2n) \times \mathbb{C}^\times, \mathbb{C}^{2n})$ with $1 \leq n$;
- (K5) $(\text{Sp}(2n) \times \text{GL}(2), \mathbb{C}^{2n} \otimes \mathbb{C}^2)$ with $2 \leq n$;
- (K6) $(\text{Sp}(2n) \times \text{GL}(3), \mathbb{C}^{2n} \otimes \mathbb{C}^3)$ with $3 \leq n$;
- (K7) $(\text{Sp}(4) \times \text{GL}(3), \mathbb{C}^4 \otimes \mathbb{C}^3)$;
- (K8) $(\text{Sp}(4) \times \text{GL}(n), \mathbb{C}^4 \otimes \mathbb{C}^n)$ with $4 \leq n$;
- (K9) $(\text{SO}(n) \times \mathbb{C}^\times, \mathbb{C}^n)$ with $3 \leq n$;
- (K10) $(\text{Spin}(10) \times \mathbb{C}^\times, \mathbb{C}^{16})$;
- (K11) $(\text{Spin}(7) \times \mathbb{C}^\times, \mathbb{C}^8)$;
- (K12) $(\text{Spin}(9) \times \mathbb{C}^\times, \mathbb{C}^{16})$;
- (K13) $(\text{G}_2 \times \mathbb{C}^\times, \mathbb{C}^7)$;
- (K14) $(\text{E}_6 \times \mathbb{C}^\times, \mathbb{C}^{27})$;
- (K15) $(\text{GL}(n) \times \mathbb{C}^\times, \wedge^2 \mathbb{C}^n \oplus \mathbb{C}^n)$ with $4 \leq n$;
- (K16) $(\text{GL}(n) \times \mathbb{C}^\times, \wedge^2 \mathbb{C}^n \oplus (\mathbb{C}^n)^*)$ with $4 \leq n$;
- (K17) $(\text{GL}(m) \times \text{GL}(n), (\mathbb{C}^m \otimes \mathbb{C}^n) \oplus \mathbb{C}^n)$ with $1 \leq m, 2 \leq n$;
- (K18) $(\text{GL}(m) \times \text{GL}(n), (\mathbb{C}^m \otimes \mathbb{C}^n) \oplus (\mathbb{C}^n)^*)$ with $1 \leq m, 2 \leq n$;
- (K19) $(\text{Sp}(2n) \times \mathbb{C}^\times \times \mathbb{C}^\times, \mathbb{C}^{2n} \oplus \mathbb{C}^{2n})$ with $2 \leq n$;

- (K20) $((\mathrm{Sp}(2n) \times \mathbb{C}^\times) \times \mathrm{GL}(2), (\mathbb{C}^{2n} \otimes \mathbb{C}^2) \oplus \mathbb{C}^2)$ with $2 \leq n$;
(K21) $(\mathrm{GL}(m) \times \mathrm{SL}(2) \times \mathrm{GL}(n), (\mathbb{C}^m \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^n))$ with $2 \leq m \leq n$;
(K22) $((\mathrm{Sp}(2m) \times \mathbb{C}^\times) \times \mathrm{SL}(2) \times \mathrm{GL}(n), (\mathbb{C}^{2m} \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^n))$ with $2 \leq m, n$;
(K23) $((\mathrm{Sp}(2m) \times \mathbb{C}^\times) \times \mathrm{SL}(2) \times (\mathrm{Sp}(2n) \times \mathbb{C}^\times), (\mathbb{C}^{2m} \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{2n}))$ with $2 \leq m, n$;
(K24) $(\mathrm{Spin}(8) \times \mathbb{C}^\times \times \mathbb{C}^\times, \mathbb{C}_+^8 \oplus \mathbb{C}_-^8)$.

Remark 3.2. The indices m and n in family (K17) and family (K18) run through a larger set than that given in Knop's List in [Kno98]. Knop communicated the revised range of indices for these families to the second author. We remark that these cases do appear in the lists of [Lea98] and [BR96]. In the family (K9) we suppose that $n \geq 3$, whereas in [Kno98] it is required that $n \geq 2$. This correction to (K9) was already made in the revised version of [Kno98] available on Knop's website.

In the rest of the present section, we will use the same notations as in [PVS12, Section 5]: in each subsection, (\overline{G}, W) will denote a member of the family from List 3.1 under consideration. Given such a spherical module (\overline{G}, W) ,

- E denotes the basis of the weight monoid of (\overline{G}, W^*) (the elements of E are called the 'basic weights' in [Kno98]);
- $V = \bigoplus_{\lambda \in E} V(\lambda)$;
- $x_0 = \sum_{\lambda \in E} v_\lambda$.

Except if stated otherwise, G will denote a connected subgroup of \overline{G} containing \overline{G}' such that (G, W) is spherical. Recall that such a group G is necessarily reductive. To lighten notation, we will use G' for the derived subgroup \overline{G}' of \overline{G} . This should not cause confusion since $(\overline{G}, \overline{G}) = (G, G) = G'$. We will use p for the projection from the weight lattice of G to the weight lattice of G' (where we fix the maximal torus $T \cap G'$ of G'). We will use $\omega, \omega', \omega''$ for weights of the first, second and third non-abelian factor of G , while ε will refer to the character $\mathbb{C}^\times \rightarrow \mathbb{C}^\times, z \mapsto z$ of \mathbb{C}^\times .

Remark 3.3. Let (\overline{G}, W) be a spherical module in Knop's List and let G be a connected subgroup of \overline{G} containing \overline{G}' . Theorem 5.1 in [Kno98] gives a criterion which characterizes, in terms of the center of G , whether (G, W) is a spherical module: (G, W) is spherical if and only if the center of G separates the weights in a certain subspace $\mathfrak{a}^* \cap \mathfrak{z}^*$ of the dual of the Lie algebra of the maximal torus of \overline{G} . The tables in [Kno98] give an explicit basis of $\mathfrak{a}^* \cap \mathfrak{z}^*$ for every spherical module (\overline{G}, W) in Knop's List.

Remark 3.4. (a) We recall from [PVS12, Remark 5.4] that the T_{ad} -weight set we obtain below for each $T_{X_0} M_S^G$ is a basis of the monoid $-w_0 \Sigma_W$, where w_0 is the longest element in the Weyl group of G (instead of $-\Sigma_W$ as in Theorem 1.1 where the T_{ad} -action from [AB05] was used).
(b) As explained in [PVS12, Remark 5.6], the computations of the T_{ad} -weight sets of $T_{X_0} M_S^G$ we perform in this section confirm Knop's computation in [Kno98, Section 5] of the "simple reflections" of the little Weyl group of the spherical modules under consideration.

3.1. **(K4) The modules $(\mathrm{Sp}(2n) \times \mathbb{C}^\times, \mathbb{C}^{2n})$ with $1 \leq n$.** Here

$$E = \{\omega_1 + \varepsilon\};$$

$$d_W = 0.$$

We will make use of the following general lemma to treat this case, as well as the cases (K9), (K11) and (K13) below.

Lemma 3.5. *Let G be a connected reductive group and let W be a spherical G -module. If E^* is the basis of the weight monoid $\mathcal{S}(W)$ of W and $x_0 = \sum_{\lambda \in E} v_\lambda \in \bigoplus_{\lambda \in E} V(\lambda)$, then $\dim W = \dim \mathfrak{g} \cdot x_0$.*

Proof. This follows from [JR09, Proposition 1.1] using the fact that $X_0 = \overline{G \cdot x_0}$ and W have the same weight monoid. \square

Applying this lemma to the modules W in the family (K4) yields the following proposition.

Proposition 3.6. *The vector space $V/\mathfrak{g} \cdot x_0$ is zero-dimensional. In particular, $\dim(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}} = d_W$. Consequently, $\dim T_{X_0} M_S^G = d_W$.*

Proof. Since $V \simeq W$ and, by Lemma 3.5, $\dim \mathfrak{g} \cdot x_0 = \dim W$, we have that $V/\mathfrak{g} \cdot x_0 = \{0\}$. \square

3.2. **(K5) The modules $(\mathrm{Sp}(2n) \times \mathrm{GL}(2), \mathbb{C}^{2n} \otimes \mathbb{C}^2)$ with $2 \leq n$.** Here

$$E = \{\omega_1 + \omega'_1, \omega_2 + \omega'_2, \omega'_2\};$$

$$d_W = 2.$$

Proposition 3.7. *The T_{ad} -module $(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$ is multiplicity-free and has T_{ad} -weight set*

$$\{\alpha_1 + \alpha', \alpha_1 + 2\delta + \alpha_n\},$$

where $\delta = 0$ if $n = 2$ and $\delta = \alpha_2 + \alpha_3 + \dots + \alpha_{n-1}$ if $n > 2$. In particular, $\dim(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}} = d_W$. Consequently, $\dim T_{X_0} M_S^G = d_W$

Proof. Note that $G' = \mathrm{Sp}(2n) \times \mathrm{SL}(2)$. Consider the G' -module $V' := V(\omega_1 + \omega'_1) \oplus V(\omega_2)$ and its element $x'_0 = v_{\omega_1 + \omega'_1} + v_{\omega_2}$. Observe that $G'_{x_0} = G'_{x'_0}$. Since $V(\omega'_2)$ is one-dimensional, we have that $(V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}} \simeq (V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$ as T_{ad} -modules.

Recall that p is the projection from the weight lattice Λ of G to the weight lattice of G' . The monoid $p(\langle E \rangle_{\mathbb{N}}) = \langle \omega_1 + \omega'_1, \omega_2 \rangle_{\mathbb{N}}$ is free and G' -saturated. By [BCF08, Theorems 3.1 and 3.10], $(V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}}$ is multiplicity-free and its T_{ad} -weights belong to Table 1 in [BCF08, page 2810]. By Proposition 2.6 the T_{ad} -weights of $(V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}}$ also belong to $\Lambda_R \cap \langle \omega_1 + \omega'_1, \omega_2 \rangle_{\mathbb{Z}}$. A straightforward computation shows that

$$(3.1) \quad \Lambda_R \cap \langle \omega_1 + \omega'_1, \omega_2 \rangle_{\mathbb{Z}} = \langle \alpha_1 + 2\delta + \alpha_n, \alpha_1 + \alpha' \rangle_{\mathbb{Z}}$$

Observe that the support of each of the two generators of $\Lambda_R \cap \langle \omega_1 + \omega'_1, \omega_2 \rangle_{\mathbb{Z}}$ in equation (3.1) contains a simple root not in the support of the other generator. Because the T_{ad} -weights of $(V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}}$ are linear combinations of the simple roots with positive coefficients, it follows that they belong to $\langle \alpha_1 + 2\delta + \alpha_n, \alpha_1 + \alpha' \rangle_{\mathbb{N}}$. One checks that for

$n > 2$, the only T_{ad} -weights in [BCF08, Table 1] satisfying this requirement are $\alpha_1 + \alpha'$ and $\alpha_1 + 2\delta + \alpha_n$.

For $n = 2$ there is a third T_{ad} -weight that satisfies it, namely $\gamma := 2\alpha_1 + 2\alpha_2$. The weight γ occurs in V' , with multiplicity one. More precisely, this T_{ad} -weight space is a line $\mathbb{C}v$ in $V(\omega_2) \subseteq V'$. We claim that $[v] \in \frac{V'}{\mathfrak{g}' \cdot x'_0}$ does not belong to $\left(\frac{V'}{\mathfrak{g}' \cdot x'_0}\right)^{G'_{x'_0}}$. We prove the claim by contradiction. Indeed, since α_2 is the only simple root such that $\gamma - \alpha_2 \in R^+ \cup \{0\}$, we have that $0 \neq [v] \in \left(\frac{V'}{\mathfrak{g}' \cdot x'_0}\right)^{G'_{x'_0}}$ would, by Proposition 2.6, imply that $X_{\alpha_2} \cdot v$ is a nonzero element of $\langle X_{-(\gamma - \alpha_2)} \cdot x'_0 \rangle_{\mathbb{C}}$. This is absurd since $X_{-(\gamma - \alpha_2)} \cdot x'_0$ has nonzero projection onto the component $V(\omega_1 + \omega'_1)$ of V' and the claim is proved.

We have shown that $(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$ is multiplicity-free and that its T_{ad} -weight set is a subset of the one in the statement of the proposition. Since Proposition 2.5(1) of [PVS12] tells us that $\dim T_{X_0} M_S^G \geq d_W$ and Corollary 2.14 of *loc.cit.* says that $T_{X_0} M_S^G \subseteq (V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$, the proposition follows. \square

3.3. (K6) The modules $(\text{Sp}(2n) \times \text{GL}(3), \mathbb{C}^{2n} \otimes \mathbb{C}^3)$ with $3 \leq n$. For these modules,

$$E = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\};$$

$$d_W = 5,$$

where

$$\begin{aligned} \lambda_1 &:= \omega_1 + \omega'_1; & \lambda_2 &:= \omega_2 + \omega'_2; & \lambda_3 &:= \omega_3 + \omega'_3; \\ \lambda_4 &:= \omega'_2; & \lambda_5 &:= \omega_1 + \omega'_3; & \lambda_6 &:= \omega_2 + \omega'_1 + \omega'_3. \end{aligned}$$

We remark that $G = \overline{G}$ is the only connected group between \overline{G}' and \overline{G} for which these modules are spherical, cf. Remark 3.3. Therefore, we assume that $G = \overline{G} = \text{Sp}(2n) \times \text{GL}(3)$ throughout this section.

In this section we will prove the following proposition.

Proposition 3.8. *The T_{ad} -module $T_{X_0} M_S^G$ is multiplicity-free. Its T_{ad} -weight set is*

$$(3.2) \quad \{\alpha_1, \alpha_2, \alpha_2 + \gamma, \alpha'_1, \alpha'_2\},$$

where $\gamma = \alpha_3$ if $n = 3$ and $\gamma = 2(\alpha_3 + \alpha_4 + \dots + \alpha_{n-1}) + \alpha_n$ if $n > 3$. In particular, $\dim T_{X_0} M_S^G = d_W$.

Proof. Let β be a T_{ad} -weight in $T_{X_0} M_S^G$. It follows from Proposition 2.19 that at least one $\lambda \in E$ has a positive coefficient in the expression of β as a \mathbb{Z} -linear combination of elements of E . Note that all elements of E except λ_3 have codimension 1. In particular, if $\lambda \neq \lambda_3$, then it follows from Proposition 2.21 that β is a simple root belonging to the set (3.2), and that its weight space has dimension one.

Since $\dim T_{X_0} M_S^G \geq d_W$ by [PVS12, Proposition 2.5(1)], what remains to prove the proposition is to show the following three claims:

Claim A: if λ_3 is the only element of E which has a positive coefficient in the expression of β , then $\beta = \alpha_2 + \gamma$ if $n > 3$ and $\beta \in \{\alpha_2 + \gamma, 2\alpha_2 + 2\gamma\}$ if $n = 3$.

Claim B: suppose $n = 3$; the T_{ad} -weight $\beta = 2\alpha_2 + 2\gamma$ does not occur in $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$, and therefore also not in the subspace $T_{X_0}M_S^G$.

Claim C: the T_{ad} -weight $\beta = \alpha_2 + \gamma$ has multiplicity at most one in $T_{X_0}M_S^G$.

We begin with Claim A. Recall from Proposition 2.6 that $\beta \in \langle E \rangle_{\mathbb{Z}} \cap \langle \Pi \rangle_{\mathbb{N}}$. Straightforward computations show that

$$(3.3) \quad \langle E \rangle_{\mathbb{Z}} \cap \Lambda_R = \langle \alpha_1, \alpha_2, \alpha'_1, \alpha'_2, \gamma \rangle_{\mathbb{Z}}$$

and that

$$\begin{aligned} \alpha_1 &= \lambda_1 + \lambda_5 - \lambda_6; \\ \alpha_2 &= \lambda_2 + \lambda_6 - \lambda_1 - \lambda_3 - \lambda_4; \\ \alpha'_1 &= \lambda_1 + \lambda_6 - \lambda_5 - \lambda_2; \\ \alpha'_2 &= \lambda_2 + \lambda_4 - \lambda_6; \\ \gamma &= 2\lambda_3 + \lambda_1 + \lambda_4 - \lambda_5 - \lambda_2 - \lambda_6. \end{aligned}$$

Let K be the basis of $\langle E \rangle_{\mathbb{Z}} \cap \Lambda_R$ given in equation (3.3). Since $\beta \in \langle \Pi \rangle_{\mathbb{N}}$ and all elements of K contain a simple root in their support, that is not in the support of any other element of K , it follows that $\beta \in \langle K \rangle_{\mathbb{N}}$. Therefore, there exist $A_1, A_2, \dots, A_5 \in \mathbb{N}$ such that

$$(3.4) \quad \beta = A_1\alpha_1 + A_2\alpha_2 + A_3\alpha'_1 + A_4\alpha'_2 + A_5\gamma.$$

From the hypothesis of Claim A, it follows that

$$(3.5) \quad \begin{cases} A_1 - A_2 + A_3 + A_5 \leq 0; \\ A_2 - A_3 + A_4 - A_5 \leq 0; \\ -A_2 + A_4 + A_5 \leq 0; \\ A_1 - A_3 - A_5 \leq 0; \\ -A_1 + A_2 + A_3 - A_4 - A_5 \leq 0. \end{cases}$$

Adding the first two inequalities in (3.5) yields that $A_1 = A_4 = 0$. Then adding the first and the last gives that $A_3 = 0$. After substituting these values into the first and last inequalities, we deduce that $A_2 = A_5$. It follows that $\beta \in \langle \alpha_2 + \gamma \rangle_{\mathbb{N}}$. Using that β is the sum of a simple root and an element of $R^+ \cup \{0\}$ (see Proposition 2.6) it follows that $\beta = \alpha_2 + \gamma$ if $n > 3$ and that $\beta \in \{\alpha_2 + \gamma, 2\alpha_2 + 2\gamma\}$ if $n = 3$. This proves Claim A.

We proceed to Claim B. Let $n = 3$ and fix $\beta = 2\alpha_2 + 2\gamma = 2\alpha_2 + 2\alpha_3$. One deduces from the well-known decompositions into T -weight spaces of $V(\omega_1)$, $V(\omega_2)$ and $V(\omega_3)$ that the T_{ad} -weight space in V of T_{ad} -weight β is a line $\mathbb{C}v$ in $V(\lambda_3) \subseteq V$. We prove Claim B by contradiction. Indeed, since α_3 is the only simple root such that $\gamma - \alpha_3 \in R^+ \cup \{0\}$, we have that $0 \neq [v] \in (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ would, by Proposition 2.6, imply that $X_{\alpha_3} \cdot v$ is a nonzero element of $\langle X_{-(\gamma-\alpha_3)} \cdot x_0 \rangle_{\mathbb{C}}$. This is absurd since $X_{-(\gamma-\alpha_3)} \cdot x_0$ has nonzero projection onto the components $V(\lambda_2)$ and $V(\lambda_6)$ of V . Claim B is proved.

Finally, we show Claim C. We fix $\beta = \alpha_2 + \gamma$. We will show that the T_{ad} -weight β has multiplicity at most one in $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$. Since $T_{X_0}M_S^G \subseteq (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$, this implies Claim C. First off, we claim that the T_{ad} -weight β only occurs in $V(\lambda_2)$, $V(\lambda_3)$ and in $V(\lambda_6)$. Indeed, β belongs to the root lattice of $\text{Sp}(2n)$, and does not occur as a T_{ad} -weight

in $V(\omega_1)$. Let Z be the subspace of $V(\lambda_2) \oplus V(\lambda_3) \oplus V(\lambda_6)$ consisting of T_{ad} -eigenvectors v of T_{ad} -weight β that satisfy the following three conditions:

$$(3.6) \quad X_{\alpha_2} \cdot v \in \langle X_{-(\beta-\alpha_2)} x_0 \rangle_{\mathbb{C}};$$

$$(3.7) \quad X_{\alpha_3} \cdot v \in \langle X_{-(\beta-\alpha_3)} x_0 \rangle_{\mathbb{C}}; \text{ and}$$

$$(3.8) \quad X_{\alpha_k} \cdot v = 0 \text{ for all } k \in \{1, 2, \dots, n\} \setminus \{2, 3\}.$$

Since $\alpha = \alpha_2$ and $\alpha = \alpha_3$ are the only simple roots such that $\beta - \alpha \in R^+ \cup \{0\}$ it follows from Proposition 2.6 that every $v \in V^\beta$ such that $0 \neq [v] \in (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ satisfies (3.6), (3.7) and (3.8). To show Claim C it is therefore enough to prove that

$$(3.9) \quad \dim Z \leq 2,$$

since the nonzero vector $X_{-\beta} \cdot x_0$ belongs to $\mathfrak{g} \cdot x_0 \cap Z$.

To prove the inequality (3.9) we will make use of the explicit description of $\mathfrak{sp}(2n)$ and its root operators given in the proof of [GW09, Theorem 2.4.1], as well as the notations therein. In particular, we have a basis $\{e_1, e_2, \dots, e_n, e_{-1}, e_{-2}, \dots, e_{-n}\}$ of \mathbb{C}^{2n} and a \mathbb{Z} -basis $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ of the weight lattice of $\text{Sp}(2n)$ such that e_i has weight ε_i and e_{-k} has weight $-\varepsilon_k$ in the defining representation of $\text{Sp}(2n)$ on \mathbb{C}^{2n} . In terms of the basis $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ of the weight lattice, the simple roots of $\text{Sp}(2n)$ are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i \in \{1, 2, \dots, n-1\}$ and $\alpha_n = 2\varepsilon_n$. Moreover, for each root δ we have a root operator $X_\delta \in \mathfrak{sp}(2n)^\delta$. In view of conditions (3.6), (3.7) and (3.8) we will make use of the root operators associated to the simple roots and to the negative roots $-(\beta - \alpha_2) = -2\varepsilon_3$ and $-(\beta - \alpha_3) = -\varepsilon_2 - \varepsilon_4$. The action of these operators on the given basis of the defining representation \mathbb{C}^{2n} of $\text{Sp}(2n)$ is as follows:

$$(3.10) \quad X_{\alpha_i} \cdot e_k = \begin{cases} e_i & \text{if } k = i+1; \\ -e_{-(i+1)} & \text{if } k = -i; \\ 0 & \text{if } k \notin \{i, -i\} \end{cases} \quad \text{for } i \in \{1, 2, \dots, n-1\};$$

$$(3.11) \quad X_{\alpha_n} \cdot e_k = \begin{cases} e_n & \text{if } k = -n; \\ 0 & \text{if } k \neq -n; \end{cases}$$

$$(3.12) \quad X_{-2\varepsilon_i} \cdot e_k = \begin{cases} e_{-i} & \text{if } k = i; \\ 0 & \text{if } k \neq i \end{cases} \quad \text{for } i \in \{1, 2, \dots, n\};$$

$$(3.13) \quad X_{-\varepsilon_i - \varepsilon_j} \cdot e_k = \begin{cases} e_{-j} & \text{if } k = i; \\ e_{-i} & \text{if } k = j; \\ 0 & \text{if } k \notin \{i, j\} \end{cases} \quad \text{where } 1 \leq i < j \leq n;$$

Note that

$$\beta = \alpha_2 + \gamma = -\omega_1 + \omega_3 = \varepsilon_2 + \varepsilon_3.$$

We now identify the weight spaces of this T_{ad} -weight in the representations $V(\omega_2)$ and $V(\omega_3)$ of $\text{Sp}(2n)$. A vector in $V(\omega_2)$ has T_{ad} -weight β if and only if it has T -weight $\omega_2 - \beta = \varepsilon_1 - \varepsilon_3$. We identify $V(\omega_2)$ with the sub- $\text{Sp}(2n)$ -representation of $\wedge^2 \mathbb{C}^{2n}$ with highest weight vector $e_1 \wedge e_2$. Then the T -weight space in $V(\omega_2)$ of weight $\varepsilon_1 - \varepsilon_3$ is the line

spanned by

$$(3.14) \quad e_1 \wedge e_{-3}.$$

A vector in $V(\omega_3)$ has T_{ad} -weight β if and only if it has T -weight $\omega_3 - \beta = \varepsilon_1$. As is well-known, $V(\omega_3)$ is the irreducible component of the $\text{Sp}(2n)$ -module $\wedge^3 \mathbb{C}^{2n}$ generated by the highest weight vector $e_1 \wedge e_2 \wedge e_3$. In the larger module $\wedge^3 \mathbb{C}^{2n}$ the T -weight space of weight ε_1 is spanned by the following vectors

$$(3.15) \quad e_1 \wedge e_2 \wedge e_{-2}, e_1 \wedge e_3 \wedge e_{-3}, \dots, e_1 \wedge e_n \wedge e_{-n}.$$

It follows from the previous paragraph that if $v \in Z$, then there exist $A_1, A_2 \in \mathbb{C}$ and $B_2, B_3, \dots, B_n \in \mathbb{C}$ such that

$$(3.16) \quad v = A_1(e_1 \wedge e_{-3} \otimes v_{\omega'_2}) + A_2(e_1 \wedge e_{-3} \otimes v_{\omega'_1 + \omega'_3}) + \sum_{k=2}^n B_k(e_1 \wedge e_k \wedge e_{-k} \otimes v_{\omega'_3}).$$

Straightforward computations using the root operators show that conditions (3.6), (3.7) and (3.8) imply that

$$(3.17) \quad A_1 = A_2 = -B_3 + B_4;$$

$$(3.18) \quad B_4 = B_5 = \dots = B_n.$$

This implies that $\dim Z \leq 3$. Note that the vector $v_1 \otimes v_{\omega'_3} \in \wedge^3 \mathbb{C}^{2n} \otimes V(\omega'_3)$, where

$$v_1 = \sum_{k=2}^n e_1 \wedge e_k \wedge e_{-k},$$

satisfies the equations (3.17) and (3.18). It is straightforward to check that v_1 is a highest weight vector. It follows that v_1 is an element of the $\text{Sp}(2n)$ -stable complement to $V(\omega_3)$ in $\wedge^3 \mathbb{C}^{2n}$. Consequently, the line spanned by $v_1 \otimes v_{\omega'_3}$ is not contained in Z , and $\dim Z \leq 2$. This proves Claim C, and the proposition. \square

3.4. (K7) The module $(\text{Sp}(4) \times \text{GL}(3), \mathbb{C}^4 \otimes \mathbb{C}^3)$. For this module we have

$$E = \{\omega_1 + \omega'_1, \omega_2 + \omega'_2, \omega'_2, \omega_1 + \omega'_3, \omega_2 + \omega'_1 + \omega'_3\}$$

$$d_W = 4.$$

Proposition 3.9. *The T_{ad} -module $T_{X_0} M_S^G$ is multiplicity-free. Its T_{ad} -weight set is*

$$(3.19) \quad \{\alpha_1, \alpha_2, \alpha'_1, \alpha'_2\}.$$

In particular, $\dim T_{X_0} M_S^G = d_W$.

Proof. Let β be a T_{ad} -weight in $T_{X_0} M_S^G$. It follows from Proposition 2.19 that at least one $\lambda \in E$ has a positive coefficient in the expression of β as a \mathbb{Z} -linear combination of elements of E . Note that all elements of E have codimension 1. It follows from Proposition 2.21 that β is a simple root and that its weight space has dimension one. Since the set (3.19) contains all simple roots of G , we can conclude that β belongs to this set and that $\dim T_{X_0} M_S^G \leq d_W$. The proposition now follows from the a priori estimate $\dim T_{X_0} M_S^G \geq d_W$, see [PVS12, Proposition 2.5(1)]. \square

3.5. **(K8) The modules** $(\mathrm{Sp}(4) \times \mathrm{GL}(n), \mathbb{C}^4 \otimes \mathbb{C}^n)$ **with** $4 \leq n$. We put $\lambda_1 = \omega_1 + \omega'_1, \lambda_2 = \omega_2 + \omega'_2, \lambda_3 = \omega'_2, \lambda_4 = \omega_1 + \omega'_3, \lambda_5 = \omega_2 + \omega'_1 + \omega'_3, \lambda_6 = \omega'_4$. Then

$$E = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\};$$

$$d_W = 5.$$

Proposition 3.10. *The T_{ad} -module $T_{X_0} M_S^G$ is multiplicity-free. Its T_{ad} -weight set is*

$$(3.20) \quad \{\alpha_1, \alpha_2, \alpha'_1, \alpha'_2, \alpha'_3\}.$$

In particular, $\dim T_{X_0} M_S^G = d_W$.

Proof. Let β be a T_{ad} -weight in $T_{X_0} M_S^G$. It follows from Proposition 2.19 that at least one $\lambda \in E$ has a positive coefficient in the expression of β as a \mathbb{Z} -linear combination of elements of E . Note that all elements of E except λ_6 have codimension 1. In particular, if $\lambda \neq \lambda_6$, then it follows from Proposition 2.21 that β is a simple root belonging to the set (3.20), and that its weight space has dimension one.

Consequently, to prove the proposition, what remains is to show that λ_6 cannot be the only element of E which has a positive coefficient in the expression of β as a linear combination of the elements of E .

Note that $\overline{G}' = \mathrm{Sp}(4) \times \mathrm{SL}(n)$ and therefore that $\dim \overline{G}/\overline{G}' = 1$. For $n = 4$ the only connected group between \overline{G}' and \overline{G} , for which W is spherical, is $G = \overline{G}$. On the other hand, if $n > 4$, then there are two such groups: $G = \overline{G}$ and $G = \overline{G}'$; cf. Remark 3.3. Straightforward computations show that

$$(3.21) \quad \langle E \rangle_{\mathbb{Z}} \cap \Lambda_R = \begin{cases} \langle \alpha_1, \alpha_2, \alpha'_1, \alpha'_2, \alpha'_3 \rangle_{\mathbb{Z}} & \text{if } G = \overline{G} \text{ or } n \text{ is odd or } n = 4; \\ \langle \alpha_1, \alpha_2, \alpha'_1, \alpha'_2, \alpha'_3, \gamma \rangle_{\mathbb{Z}} & \text{if } G = \overline{G}' \text{ and } n \text{ is even and } n > 4 \end{cases}$$

where

$$(3.22) \quad \gamma = (n-4)\alpha'_4 + (n-5)\alpha'_5 + \dots + 2\alpha'_{n-2} + \alpha'_{n-1}.$$

Let K be the basis of $\langle E \rangle_{\mathbb{Z}} \cap \Lambda_R$ in equation (3.21). Since $\beta \in \langle \Pi \rangle_{\mathbb{N}}$ and all elements of K contain a simple root in their support, which is not in the support of any other element of K , it follows that $\beta \in \langle K \rangle_{\mathbb{N}}$. We have the following equalities:

$$\begin{aligned} \alpha_1 &= \lambda_1 + \lambda_4 - \lambda_5; \\ \alpha_2 &= \lambda_2 + \lambda_5 - \lambda_3 - \lambda_1 - \lambda_4; \\ \alpha'_1 &= \lambda_1 + \lambda_5 - \lambda_4 - \lambda_2; \\ \alpha'_2 &= \lambda_2 + \lambda_3 - \lambda_5; \\ \alpha'_3 &= \lambda_4 + \lambda_5 - \lambda_6 - \lambda_1 - \lambda_2; \\ \gamma &= (n-3)\lambda_6 - \frac{n-4}{2}[\lambda_4 + \lambda_5 - \lambda_1 - \lambda_2 + \lambda_3]. \end{aligned}$$

As one easily sees, γ is the only element of K in which λ_6 has a positive coefficient. Consequently, the Proposition follows from equation (3.21) for $G = \overline{G}$, for odd n and for $n = 4$.

We now assume that $G = \overline{G}'$, that n is even and at least 6 and that λ_6 has a positive coefficient in β . We will come to a contradiction. Our assumptions imply that γ has

a positive coefficient in the expression of β as an \mathbb{N} -linear combination of the elements of K . Recall from Proposition 2.6 that β is the sum of a simple root and an element of $R^+ \cup \{0\}$. By equation (3.22) this is only possible if $n = 6$ and if β is one of the following three elements of $\langle K \rangle_{\mathbb{N}}$:

$$\begin{aligned}\beta_1 &:= \alpha'_1 + \alpha'_2 + \alpha'_3 + \gamma; \\ \beta_2 &:= \alpha'_2 + \alpha'_3 + \gamma; \\ \beta_3 &:= \alpha'_3 + \gamma.\end{aligned}$$

Since $\beta_1 = \lambda_1 + 2\lambda_6 - \lambda_4$ and $\beta_2 = \lambda_2 + 2\lambda_6 - \lambda_5$, it follows that β cannot be either of them by Proposition 2.20. Finally, $\beta = \beta_3$ is not possible because if $v \in V^{\beta_3}$ with $0 \neq [v] \in (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$, then it follows from Proposition 2.6 that $X_{\alpha'_4} \cdot v$ is a nonzero element in $\langle X_{-(\beta_3 - \alpha'_4)} \cdot x_0 \rangle_{\mathbb{C}}$, since α'_4 is the only simple root such that $\beta_3 - \alpha'_4 \in R^+ \cup \{0\}$. Since $X_{-(\beta_3 - \alpha'_4)} \cdot x_0$ has nonzero projection on $V(\lambda_4)$, so does v , but β_3 does not occur as a T_{ad} -weight in $V(\lambda_4)$ as follows immediately from the well known list of T -weights in $V(\omega'_3)$. This completes the proof. \square

3.6. (K9) The modules $(\text{SO}(n) \times \mathbb{C}^\times, \mathbb{C}^n)$ with $3 \leq n$. For these modules

$$\begin{aligned}E &= \{\omega_1 + \varepsilon, 2\varepsilon\}; \\ d_W &= 1.\end{aligned}$$

Proposition 3.11. *The T_{ad} -module $(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$ is one-dimensional. Its weight is*

$$\begin{aligned}2\alpha_1 + 2\alpha_2 + \dots + 2\alpha_{(n/2)-2} + \alpha_{(n/2)-1} + \alpha_{n/2} & \quad \text{if } n \text{ is even;} \\ 2\alpha_1 + 2\alpha_2 + \dots + 2\alpha_{(n-1)/2} & \quad \text{if } n \text{ is odd.}\end{aligned}$$

In particular, $\dim(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}} = d_W$. Consequently, $\dim T_{X_0} M_S^G = d_W$.

Proof. Observe that $\dim V = \dim W + 1$, since $V(2\varepsilon)$ is one-dimensional. Since $\dim W = \dim \mathfrak{g} \cdot x_0$ (by Lemma 3.5), this implies that $\dim V/\mathfrak{g} \cdot x_0 = 1$. Since $d_W = 1$, this implies that $\dim(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}} = 1$ by [PVS12, Proposition 2.5(1) and Corollary 2.14].

We now find the T_{ad} -weight of $(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$. Using the well-known T -weight space decomposition of the G -module $V(\omega_1 + \varepsilon) \simeq \mathbb{C}^n \otimes \mathbb{C}_\varepsilon$, the fact that $V(2\varepsilon) \subset \mathfrak{g} \cdot x_0$ and [PVS12, Lemma 2.16(4)], one readily checks that the one-dimensional T_{ad} -module $V/\mathfrak{g} \cdot x_0$ has the T_{ad} -weight given in the proposition. \square

3.7. (K10) The module $(\text{Spin}(10) \times \mathbb{C}^\times, \mathbb{C}^{16})$. Here

$$\begin{aligned}E &= \{\omega_5 + \varepsilon, \omega_1 + 2\varepsilon\}; \\ d_W &= 1.\end{aligned}$$

Proposition 3.12. *The T_{ad} -module $(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$ is one-dimensional. Its weight is*

$$\alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5.$$

In particular, $\dim(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}} = d_W$. Consequently, $\dim T_{X_0} M_S^G = d_W$.

Proof. Recall that p is the projection from the weight lattice Λ of G to the weight lattice of $G' = \text{Spin}(10)$. We first observe that $W = V(\omega_5 + \varepsilon)$ is spherical for $G' = \text{Spin}(10)$ (cf. Remark 3.3) and that its weight monoid $p(\mathcal{S})$ is G' -saturated. By [PVS12, Corollary 2.27] it follows that $\dim(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}} = d_W$.

While we do not need it for Theorem 1.2, we give a proof of the claim that the T_{ad} -weight of $(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$ is $\alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5$. A straightforward calculation shows that

$$(3.23) \quad p(\langle E \rangle_{\mathbb{Z}}) \cap \Lambda_R = \langle 2\alpha_1 + \alpha_2 - \alpha_5, -2\alpha_1 + 2\alpha_3 + \alpha_4 + 3\alpha_5 \rangle_{\mathbb{Z}}$$

$$(3.24) \quad = \langle \beta_1, \beta_2 \rangle_{\mathbb{Z}}.$$

where $\beta_1 = \alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5$ and $\beta_2 = 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5$. If β is a T_{ad} -weight occurring in $(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$, then, by Proposition 2.6,

$$(3.25) \quad \beta \in \langle \Pi \rangle_{\mathbb{N}} \cap p(\langle E \rangle_{\mathbb{Z}}) = \langle \Pi \rangle_{\mathbb{N}} \cap \langle \beta_1, \beta_2 \rangle_{\mathbb{Z}}; \text{ and}$$

$$(3.26) \quad \beta \in \Pi + (R^+ \cup \{0\})$$

In the root system of type D_5 , if an element of $\Pi + (R^+ \cup \{0\})$ is written as a linear combination of the simple roots, then none of the coefficients are greater than 3. Consequently, (3.25) and (3.26) imply that there exists $a, b \in \mathbb{Z}$ such that $\beta = a\beta_1 + b\beta_2$ and

$$(3.27) \quad 3 \geq 2b \geq 0$$

$$(3.28) \quad 3 \geq 2a + 2b \geq 0$$

$$(3.29) \quad 3 \geq 2a + b \geq 0$$

It follows from (3.27) that $b \in \{0, 1\}$. If $b = 0$, then it follows from (3.28) that $a \in \{0, 1\}$. If $b = 1$, then it follows from (3.29) that $a \in \{0, 1\}$, and then (3.28) implies that $a = 0$. Since $\beta \neq 0$, we have shown that $\beta = \beta_1$ or $\beta = \beta_2$.

To finish the proof, we have to show that β_2 cannot occur as a T_{ad} -weight in $(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$. To get a contradiction, suppose that $v \in V^{\beta_2}$ such that $0 \neq [v] \in (V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$. Since $\alpha = \alpha_1$ is the only simple root such that $\beta_2 - \alpha \in R^+ \cup \{0\}$, it follows from Proposition 2.6 that

$$(3.30) \quad X_{\alpha_1} \cdot v \in \langle X_{-(\beta_2 - \alpha_1)} \cdot x_0 \rangle_{\mathbb{C}} \setminus \{0\}$$

Since $\langle (\beta_2 - \alpha_1)^\vee, \omega_1 \rangle \neq 0$ and $\langle (\beta_2 - \alpha_1)^\vee, \omega_5 \rangle \neq 0$, the vector $X_{-(\beta_2 - \alpha_1)} \cdot x_0$ has nonzero projection on both irreducible components of V . This is in contradiction with (3.30), since the T_{ad} -weight β does not occur in $V(\omega_5)$. This finishes the proof. \square

Remark 3.13. The fact that the T_{ad} -weight of $(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$ is $\alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5$, which is equal to $\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5$, can also be deduced from the description of the little Weyl group of the module W^* given in [Kno98] (see [PVS12, Remark 2.8] for some context.)

3.8. (K11) The module $(\text{Spin}(7) \times \mathbb{C}^\times, \mathbb{C}^8)$. Here

$$E = \{\omega_3 + \varepsilon, 2\varepsilon\};$$

$$d_W = 1.$$

Proposition 3.14. *The T_{ad} -module $(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$ is one-dimensional. Its weight is*

$$\alpha_1 + 2\alpha_2 + 3\alpha_3.$$

In particular, $\dim(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}} = d_W$. Consequently, $\dim T_{X_0} M_S^G = d_W$

Proof. The proof that $\dim(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}} = 1$ is exactly like in the proof of Proposition 3.11. We now find the T_{ad} -weight of $(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$, also like in the proof of Proposition 3.11. Using the T -weight space decomposition of the G -module $V(\omega_3 + \varepsilon)$ (which can be computed by hand or with *LiE* [vLCL92]), the fact that $V(2\varepsilon) \subset \mathfrak{g} \cdot x_0$ and [PVS12, Lemma 2.16(4)], one readily checks that the one-dimensional T_{ad} -module $V/\mathfrak{g} \cdot x_0$ has the T_{ad} -weight given in the proposition. \square

3.9. (K12) The module $(\text{Spin}(9) \times \mathbb{C}^\times, \mathbb{C}^{16})$. Here

$$E = \{\omega_4 + \varepsilon, \omega_1 + 2\varepsilon, 2\varepsilon\};$$

$$d_W = 2.$$

Proposition 3.15. *The T_{ad} -module $(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$ is multiplicity-free and has T_{ad} -weight set*

$$\{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + 2\alpha_3 + 3\alpha_4\}.$$

In particular, $\dim(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}} = d_W$. Consequently, $\dim T_{X_0} M_S^G = d_W$

Proof. Observe that $G' = \text{Spin}(9)$. Consider the G' -module $V' := V(\omega_1) \oplus V(\omega_4)$ and its element $x'_0 = v_{\omega_1} + v_{\omega_4}$. Observe that $G'_{x_0} = G'_{x'_0}$. Since $V(2\varepsilon)$ is one-dimensional, we have that $(V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}} \simeq (V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$ as T_{ad} -modules. Put $E' = p(E) = \{\omega_1, \omega_4\}$.

Let β be a T_{ad} -weight of $(V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}}$. Then, by Proposition 2.6,

$$(3.31) \quad \beta \in \langle E' \rangle_{\mathbb{Z}} \cap \langle \Pi \rangle_{\mathbb{N}}$$

$$(3.32) \quad \beta \in \Pi + (R^+ \cup \{0\})$$

A straightforward computation shows that

$$(3.33) \quad \langle E' \rangle_{\mathbb{Z}} \cap \langle \Pi \rangle_{\mathbb{Z}} = \langle \beta_1, \beta_2 \rangle_{\mathbb{Z}},$$

where $\beta_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ and $\beta_2 = \alpha_2 + 2\alpha_3 + 3\alpha_4$. The explicit description of R^+ for $\text{Spin}(9)$ (see, e.g. [Bou68, Planche II]) shows that (3.32) implies that if β is written as a linear combination of the simple roots, then the coefficient of α_1 is at most 2 and that of the other simple roots is at most 3. Combined with (3.31) and (3.33) this implies that there exist $a, b \in \mathbb{Z}$ such that $\beta = a\beta_1 + b\beta_2$ and

$$2 \geq a \geq 0; \text{ and}$$

$$3 \geq a + 3b \geq 0$$

This system implies that $(a, b) \in \{(0, 0), (1, 0), (2, 0), (0, 1)\}$. Since $\beta \neq 0$, we have shown that

$$(3.34) \quad \beta \in \{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, 2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4), \alpha_2 + 2\alpha_3 + 3\alpha_4\}.$$

We claim that $\beta_3 := 2(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$ does not occur as a T_{ad} -weight in $(V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}}$. We will argue by contradiction; assume $v \in V'$ is a T_{ad} -eigenvector of weight β_3 such that $[v]$ is nonzero in $(V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}}$. Using the explicit description of R^+ once more,

one readily checks that $\alpha = \alpha_1$ is the only simple root α such that $\beta_3 - \alpha \in R^+ \cup \{0\}$. By Proposition 2.6 this implies that

$$(3.35) \quad X_{\alpha_1} \cdot v \in \langle X_{-(\beta_3 - \alpha_1)} \cdot x'_0 \rangle_{\mathbb{C}} \setminus \{0\}.$$

Because $\langle (\beta_3 - \alpha_1)^\vee, \omega_1 \rangle \neq 0$ and $\langle (\beta_3 - \alpha_1)^\vee, \omega_4 \rangle \neq 0$, we have that $X_{-(\beta_3 - \alpha_1)} x'_0$ has nonzero projection to both summands $V(\omega_1)$ and $V(\omega_4)$ of V' . On the other hand, the following computation in *LiE* shows that β_3 does not occur as a T_{ad} -weight in $V(\omega_4)$.

```

setdefault(B4)
omega4=[0,0,0,1]
beta3=[2,0,0,0]
Demazure(omega4)|(omega4-beta3)
-- output: 0

```

This implies that v is in the kernel of the projection onto the summand $V(\omega_4)$ of V' , which is in contradiction with equation (3.35). This proves the claim.

Since the monoid $\langle E' \rangle_{\mathbb{N}}$ is free and G' -saturated, we know that the T_{ad} -module $(V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}}$ is multiplicity-free by [BCF08, Theorem 3.10]. Equation (3.34) and the claim above then imply that $\dim(V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}} \leq 2$. Since $d_W = 2$ this proves the proposition, because $T_{X_0} M_S^G \subseteq (V/\mathfrak{g} \cdot x_0)^{G_{x_0}} \subseteq (V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}}$ and $\dim T_{X_0} M_S^G \geq d_W$, by [PVS12, Proposition 2.5(1)]. \square

3.10. (K13) The module $(G_2 \times \mathbb{C}^\times, \mathbb{C}^7)$. Here

$$E = \{\omega_1 + \varepsilon, 2\varepsilon\};$$

$$d_W = 1.$$

Proposition 3.16. *The T_{ad} -module $(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$ is one-dimensional. Its weight is*

$$4\alpha_1 + 2\alpha_2.$$

In particular, $\dim(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}} = d_W$. Consequently, $\dim T_{X_0} M_S^G = d_W$

Proof. Same argument as for Proposition 3.14. \square

3.11. (K14) The module $(E_6 \times \mathbb{C}^\times, \mathbb{C}^{27})$. Here

$$E = \{\omega_1 + \varepsilon, \omega_6 + 2\varepsilon, 3\varepsilon\};$$

$$d_W = 2.$$

Proposition 3.17. *The T_{ad} -module $(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$ is multiplicity-free and has T_{ad} -weight set*

$$\{\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6, 2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5\}.$$

In particular, $\dim(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}} = d_W$. Consequently, $\dim T_{X_0} M_S^G = d_W$

Proof. Note that $G' = E_6$. Consider the G' -module $V' := V(\omega_1) \oplus V(\omega_6)$ and its element $x'_0 = v_{\omega_1} + v_{\omega_6}$. Since $V(3\varepsilon) \subseteq \mathfrak{g} \cdot x_0$, we have that $(V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}} \simeq (V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$ as T_{ad} -modules. Observe that $G'_{x_0} = G'_{x'_0}$.

The monoid $p(\langle E \rangle_{\mathbb{N}}) = \langle \omega_1, \omega_6 \rangle_{\mathbb{N}}$ is free and G' -saturated. By [BCF08, Theorems 3.1 and 3.10], $(V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}}$ is multiplicity-free and its T_{ad} -weights belong to Table 1 in

[BCF08, page 2810]. By Proposition 2.6 the T_{ad} -weights of $(V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}}$ also belong to $\Lambda_R \cap \langle \omega_1, \omega_6 \rangle_{\mathbb{Z}}$. A straightforward computation shows that

$$(3.36) \quad \Lambda_R \cap \langle \omega_1, \omega_6 \rangle_{\mathbb{Z}} = \langle \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6, 2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 \rangle_{\mathbb{Z}}$$

Observe that the support of each of the two generators of $\Lambda_R \cap \langle \omega_1, \omega_6 \rangle_{\mathbb{Z}}$ in equation (3.36) contains a simple root not in the support of the other generator. Because the T_{ad} -weights of $(V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}}$ belong to $\langle \Pi \rangle_{\mathbb{N}}$, it follows that they belong to $\langle \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6, 2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 \rangle_{\mathbb{N}}$. Because none of the T_{ad} -weights in [BCF08, Table 1] supported on a subdiagram of E_6 has a coefficient greater than 2, it follows that the T_{ad} -weights of $(V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}}$ are a subset of $\{\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6, 2\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5\}$. Since $d_W = 2$, this proves the proposition. \square

3.12. (K19) The module $(\text{Sp}(2n) \times \mathbb{C}^\times \times \mathbb{C}^\times, \mathbb{C}^{2n} \oplus \mathbb{C}^{2n})$ with $2 \leq n$. For these modules,

$$E = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\};$$

$$d_W = 2,$$

where

$$\lambda_1 := \omega_1 + \varepsilon;$$

$$\lambda_2 := \omega_1 + \varepsilon';$$

$$\lambda_3 := \omega_2 + \varepsilon + \varepsilon';$$

$$\lambda_4 := \varepsilon + \varepsilon'.$$

Note that $G = \overline{G}$ is the only connected group between \overline{G}' and \overline{G} for which this module is spherical, cf. Remark 3.3. Therefore, we can assume throughout this section that $G = \overline{G} = \text{Sp}(2n) \times \mathbb{C}^\times \times \mathbb{C}^\times$.

Proposition 3.18. *The T_{ad} -module $T_{X_0}M_S^G$ is multiplicity-free and has T_{ad} -weight set*

$$\{\alpha_1, \alpha_1 + \gamma\},$$

where $\gamma = 2(\alpha_2 + \alpha_3 + \dots + \alpha_{n-1}) + \alpha_n$ if $n > 2$ and $\gamma = \alpha_2$ if $n = 2$. In particular, $\dim T_{X_0}M_S^G = d_W$.

Proof. This proof is similar to that of Proposition 3.8. Let β be a T_{ad} -weight in $T_{X_0}M_S^G$. By Proposition 2.6, we know that $\beta \in \langle \Pi \rangle_{\mathbb{N}} \cap \langle E \rangle_{\mathbb{Z}}$. A straightforward computation shows that $\Lambda_R \cap \langle E \rangle_{\mathbb{Z}} = \langle \alpha_1, \gamma \rangle_{\mathbb{Z}}$. Since α_1 is not in the support of γ and α_2 is in the support of γ but not in the support of α_1 , it follows that

$$(3.37) \quad \beta \in \langle \alpha_1, \gamma \rangle_{\mathbb{N}}.$$

By Proposition 2.19, at least one element of $\{\lambda_1, \lambda_2, \lambda_3\}$ must have a positive coefficient in the expression of β as a linear combination of the elements of E . Since λ_1 and λ_2 have codimension 1, it follows from Proposition 2.21 that if one of them has a positive coefficient, then $\beta = \alpha_1$ and β has multiplicity one in $T_{X_0}M_S^G$. To finish the proof it is therefore enough to show the following four claims:

Claim A: if λ_3 is the only element of E which has a positive coefficient in the expression of β as a linear combination of the elements of E , then $\beta \in \{\alpha_1 + \gamma, \gamma\}$ if $n > 2$ and $\beta \in \{\alpha_1 + \gamma, \gamma, 2\alpha_1 + 2\gamma\}$ if $n = 2$.

Claim B: the T_{ad} -weight $\beta = \gamma$ does not occur in $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$, and therefore also not in the subspace $T_{X_0}M_S^G$.

Claim C: suppose $n = 2$; the T_{ad} -weight $\beta = 2\alpha_1 + 2\gamma$ does not occur in $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$, and therefore also not in the supspace $T_{X_0}M_S^G$.

Claim D: the T_{ad} -weight $\beta = \alpha_1 + \gamma$ has multiplicity at most one in $T_{X_0}M_S^G$.

To prove Claim A, we first observe that

$$\begin{aligned}\alpha_1 &= \lambda_1 + \lambda_2 - \lambda_3; \\ \gamma &= 2\lambda_3 - \lambda_1 - \lambda_2 - \lambda_4.\end{aligned}$$

It follows from (3.37) and the hypothesis of Claim A, that there exist $A, B \in \mathbb{N}$ with $B > 0$ and $B \geq A$ such that

$$(3.38) \quad \beta = A\alpha_1 + B\gamma.$$

For $n > 2$ the only β as in (3.38) that satisfy Proposition 2.6(a) are γ and $\alpha_1 + \gamma$. For $n = 2$, there are three additional such β , namely $\beta = 2\gamma = 2\alpha_2$, $\beta = \alpha_1 + 2\gamma = \alpha_1 + 2\alpha_2$ and $\beta = 2\alpha_1 + 2\alpha_2$. Proposition 2.20 tells us that $\beta = 2\alpha_2$ and $\beta = \alpha_1 + 2\alpha_2$ cannot occur as a T_{ad} -weight in $T_{X_0}M_S^G$. This finishes the proof of Claim A.

We proceed to Claim B. Let $\beta = \gamma$. Observe that the T_{ad} -weight γ does not occur in the G -modules $V(\lambda_1)$, $V(\lambda_2)$ and $V(\lambda_4)$. It occurs in $V(\lambda_3)$ with multiplicity one: the T_{ad} -weight space in $V(\lambda_3)$ of weight β is spanned by $X_{-\beta}v_{\lambda_3}$. It follows that β occurs with multiplicity one in V and that its weight space is a subspace of $\mathfrak{g} \cdot x_0$. This proves Claim B.

We move to Claim C. Let $n = 2$ and $\beta = 2\alpha_1 + 2\alpha_2$. One deduces from the well-known decompositions into T -weight spaces of $V(\omega_1)$ and $V(\omega_2)$ that the T_{ad} -weight space in V of T_{ad} -weight β is a line $\mathbb{C}v$ in $V(\lambda_3) \subseteq V$. We prove Claim C by contradiction. Indeed, since α_2 is the only simple root such that $\beta - \alpha_2 \in R^+ \cup \{0\}$, we have that $0 \neq [v] \in (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ would, by Proposition 2.6, imply that $X_{\alpha_2} \cdot v$ is a nonzero element of $\langle X_{-(\beta-\alpha_2)} \cdot x_0 \rangle_{\mathbb{C}}$. This is absurd since $X_{-(\beta-\alpha_2)} \cdot x_0$ has nonzero projection onto the components $V(\lambda_1)$ and $V(\lambda_2)$ of V . Claim C is proved.

Finally, we show Claim D. We fix $\beta = \alpha_1 + \gamma$. We will show that the T_{ad} -weight β has multiplicity at most one in $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$. Since $T_{X_0}M_S^G \subseteq (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$, this implies Claim D. First off, we observe that the T_{ad} -weight β can only occur in $V(\lambda_1)$, $V(\lambda_2)$ or $V(\lambda_3)$. Let Z be the subspace of $V(\lambda_1) \oplus V(\lambda_2) \oplus V(\lambda_3)$ consisting of T_{ad} -eigenvectors v of T_{ad} -weight β that satisfy the following three conditions:

$$(3.39) \quad X_{\alpha_1} \cdot v \in \langle X_{-(\beta-\alpha_1)}x_0 \rangle_{\mathbb{C}};$$

$$(3.40) \quad X_{\alpha_2} \cdot v \in \langle X_{-(\beta-\alpha_2)}x_0 \rangle_{\mathbb{C}}; \text{ and}$$

$$(3.41) \quad X_{\alpha_k} \cdot v = 0 \text{ for all } k \in \{1, 2, \dots, n\} \setminus \{1, 2\}.$$

By Proposition 2.6, every $v \in V^\beta$ with $0 \neq [v] \in (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ satisfies (3.39), (3.40) and (3.41). To show Claim D it is therefore enough to prove that

$$(3.42) \quad \dim Z \leq 2,$$

since the nonzero vector $X_{-\beta} \cdot x_0$ belongs to $\mathfrak{g} \cdot x_0 \cap Z$.

To prove the inequality (3.42) we will make use of the explicit description of $\mathfrak{sp}(2n)$ and its root operators given in the proof of [GW09, Theorem 2.4.1], as well as the notations therein like we did in the proof of Proposition 3.8; see page 21.

Note that

$$\beta = \alpha_1 + \gamma = \omega_2 = \varepsilon_1 + \varepsilon_2.$$

We now identify the weight spaces of this T_{ad} -weight in the representations $V(\omega_1)$ and $V(\omega_2)$ of $\text{Sp}(2n)$. A vector in $V(\omega_1)$ has T_{ad} -weight β if and only if it has T -weight $\omega_1 - \beta = -\varepsilon_2$. We identify $V(\omega_1)$ with the standard representation \mathbb{C}^{2n} of $\text{Sp}(2n)$, which has e_1 as a highest weight vector. Then the T -weight space of weight $-\varepsilon_2$ is the line spanned by e_{-2} .

A vector in $V(\omega_2)$ has T_{ad} -weight β if and only if it has T -weight $\omega_2 - \beta = 0$. As is well-known, $V(\omega_2)$ is the irreducible component of the $\text{Sp}(2n)$ -module $\wedge^2 \mathbb{C}^{2n}$ generated by the highest weight vector $e_1 \wedge e_2$. In the larger module $\wedge^2 \mathbb{C}^{2n}$ the T -weight space of weight 0 is spanned by the following vectors

$$(3.43) \quad e_1 \wedge e_{-1}, e_2 \wedge e_{-2}, \dots, e_n \wedge e_{-n}.$$

It follows from the previous paragraph that if $v \in Z$, then there exist $A, B \in \mathbb{C}$ and $C_1, C_2, \dots, C_n \in \mathbb{C}$ such that

$$(3.44) \quad v = A(e_{-2} \otimes v_\varepsilon) + B(e_{-2} \otimes v_{\varepsilon'}) + \sum_{k=1}^n C_k(e_k \wedge e_{-k} \otimes v_{\varepsilon+\varepsilon'}).$$

Straightforward computations using the root operators show that conditions (3.39), (3.40) and (3.41) imply that when $n > 2$,

$$(3.45) \quad A = B = C_3 - C_2; \text{ and}$$

$$(3.46) \quad C_3 = C_4 = \dots = C_n,$$

and that when $n = 2$,

$$(3.47) \quad A = B.$$

Either way, this implies that $\dim Z \leq 3$. Note that the vector $v_1 \otimes v_{\varepsilon+\varepsilon'} \in \wedge^2 \mathbb{C}^{2n} \otimes V(\varepsilon + \varepsilon')$, where

$$v_1 = \sum_{k=1}^n e_k \wedge e_{-k},$$

satisfies the equations (3.45) and (3.46). It is straightforward to check that v_1 is a highest weight vector. It follows that v_1 is an element of the $\text{Sp}(2n)$ -stable complement to $V(\omega_2)$ in $\wedge^2 \mathbb{C}^{2n}$. Consequently, the line spanned by $v_1 \otimes v_{\varepsilon+\varepsilon'}$ is not contained in Z , and $\dim Z \leq 2$. This proves Claim D, and the proposition. \square

Remark 3.19. Proceeding as in the proof of [PVS12, Proposition 5.14], one can show that in fact $(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$ is a multiplicity-free T_{ad} -module with the same T_{ad} -weight set as $T_{X_0} M_S$.

3.13. **(K20) The modules** $((\mathrm{Sp}(2n) \times \mathbb{C}^\times) \times \mathrm{GL}(2), (\mathbb{C}^{2n} \otimes \mathbb{C}^2) \oplus \mathbb{C}^2)$ **with** $2 \leq n$. For these modules,

$$E = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\};$$

$$d_W = 3,$$

where

$$\begin{aligned} \lambda_1 &:= \omega'_1; & \lambda_2 &:= \omega_1 + \varepsilon + \omega'_1; & \lambda_3 &:= \omega_1 + \varepsilon + \omega'_2; \\ \lambda_4 &:= \omega_2 + 2\varepsilon + \omega'_2; & \lambda_5 &:= 2\varepsilon + \omega'_2. \end{aligned}$$

We remark that $G = \overline{G}$ is the only connected group between \overline{G}' and \overline{G} for which these modules are spherical, cf. Remark 3.3. Therefore, we assume $G = \overline{G} = \mathrm{Sp}(2n) \times \mathbb{C}^\times \times \mathrm{GL}(2)$ throughout this section.

In this section we will prove the following proposition.

Proposition 3.20. *The T_{ad} -module $T_{X_0}M_S^G$ is multiplicity-free. Its T_{ad} -weight set is*

$$(3.48) \quad \{\alpha_1, \alpha_1 + \gamma, \alpha'_1\},$$

where $\gamma_1 = \alpha_2$ if $n = 2$ and $\gamma = 2(\alpha_2 + \alpha_3 + \dots + \alpha_{n-1}) + \alpha_n$ if $n > 2$. In particular, $\dim T_{X_0}M_S^G = d_W$.

Proof. The argument is very similar to that of Proposition 3.8. Let β be a T_{ad} -weight in $T_{X_0}M_S^G$. It follows from Proposition 2.19 that at least one $\lambda \in E$ has a positive coefficient in the expression of β as a \mathbb{Z} -linear combination of elements of E . Note that all elements of E except λ_4 have codimension 1. In particular, if $\lambda \neq \lambda_4$, then it follows from Proposition 2.21 that β is a simple root belonging to the set (3.48), and that its weight space has dimension one. Consequently, to prove the proposition, what remains is to show the following two claims:

Claim A: if λ_4 is the only element of E which has a positive coefficient in the expression of β , then $\beta = \alpha_1 + \gamma$.

Claim B: the T_{ad} -weight $\beta = \alpha_1 + \gamma$ has multiplicity at most one in $T_{X_0}M_S^G$.

We begin with Claim A. Recall from Proposition 2.6 that $\beta \in \langle E \rangle_{\mathbb{Z}} \cap \langle \Pi \rangle_{\mathbb{N}}$. Straightforward computations show that

$$(3.49) \quad \langle E \rangle_{\mathbb{Z}} \cap \Lambda_R = \langle \alpha_1, \alpha'_1, \gamma \rangle_{\mathbb{Z}}$$

and that

$$\begin{aligned} \alpha_1 &= \lambda_2 + \lambda_3 - \lambda_1 - \lambda_4; \\ \alpha'_1 &= \lambda_1 + \lambda_2 - \lambda_3; \\ \gamma &= 2\lambda_4 + \lambda_1 - \lambda_2 - \lambda_3 - \lambda_5. \end{aligned}$$

Let K be the basis of $\langle E \rangle_{\mathbb{Z}} \cap \Lambda_R$ given in equation (3.49). Since $\beta \in \langle \Pi \rangle_{\mathbb{N}}$ and all elements of K contain a simple root in their support, which is not in the support of any other element of K , it follows that $\beta \in \langle K \rangle_{\mathbb{N}}$. Therefore, there exist $A, B, C \in \mathbb{N}$ such that

$$(3.50) \quad \beta = A\alpha_1 + B\alpha'_1 + C\gamma.$$

From the hypothesis of Claim A, it follows that

$$(3.51) \quad \begin{cases} -A + B + C \leq 0; \\ A + B - C \leq 0; \\ A - B - C \leq 0. \end{cases}$$

Adding the first two inequalities in (3.51) yields that $B = 0$. After substituting $B = 0$, the first two inequalities yield that $A = C$. It follows that $\beta \in \langle \alpha_1 + \gamma \rangle_{\mathbb{N}}$. Using that β is the sum of a simple root and an element of $R^+ \cup \{0\}$ (see Proposition 2.6) it follows that $\beta = \alpha_1 + \gamma$ when $n > 2$ and that $\beta \in \{\alpha_1 + \gamma, 2\alpha_1 + 2\gamma\}$ when $n = 2$. With an argument like that for Claim C in the proof of Proposition 3.18, one shows that $2\alpha_1 + 2\gamma$ cannot occur as a T_{ad} -weight in $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ when $n = 2$. This proves Claim A.

The argument for Claim B is the same as that for Claim D in Proposition 3.18 above. \square

3.14. (K22) The modules $((\text{Sp}(2m) \times \mathbb{C}^\times) \times \text{SL}(2) \times \text{GL}(n), (\mathbb{C}^{2m} \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^n))$ **with** $2 \leq m, n$. For these modules,

$$E = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6\};$$

$$d_W = 4,$$

where

$$\begin{aligned} \lambda_1 &:= \omega_1 + \varepsilon + \omega'; & \lambda_2 &:= \omega' + \omega''_1; & \lambda_3 &:= \omega_1 + \varepsilon + \omega''_1; \\ \lambda_4 &:= \omega_2 + 2\varepsilon; & \lambda_5 &:= \omega''_2; & \lambda_6 &:= 2\varepsilon. \end{aligned}$$

In this section we will prove the following proposition.

Proposition 3.21. *The T_{ad} -module $T_{X_0}M_S^G$ is multiplicity-free. Its T_{ad} -weight set is*

$$(3.52) \quad \{\alpha_1, \alpha_1 + \gamma, \alpha', \alpha''_1\},$$

where $\gamma = \alpha_2$ if $m = 2$ and $\gamma = 2(\alpha_2 + \alpha_3 + \dots + \alpha_{m-1}) + \alpha_m$ if $m > 2$. In particular, $\dim T_{X_0}M_S^G = d_W$.

Proof. The argument is very similar to that of Propositions 3.8, 3.18 and 3.20. Let β be a T_{ad} -weight in $T_{X_0}M_S^G$. It follows from Proposition 2.19 that at least one $\lambda \in E$ has a positive coefficient in the expression of β as a \mathbb{Z} -linear combination of elements of E . Note that all elements of E except λ_4 and λ_5 have codimension 1. In particular, if $\lambda \notin \{\lambda_4, \lambda_5\}$, then it follows from Proposition 2.21 that β is a simple root belonging to the set (3.52), and that its weight space has dimension one.

Consequently, to prove the proposition, what remains is to show the following two claims:

Claim A: if λ_4 or λ_5 are the only elements of E which have a positive coefficient in the expression of β , then $\beta = \alpha_1 + \gamma$.

Claim B: the T_{ad} -weight $\beta = \alpha_1 + \gamma$ has multiplicity at most one in $T_{X_0}M_S^G$.

We begin with Claim A. Recall from Proposition 2.6 that $\beta \in \langle E \rangle_{\mathbb{Z}} \cap \langle \Pi \rangle_{\mathbb{N}}$. Consequently $\beta \in p(\langle E \rangle_{\mathbb{Z}}) \cap \Lambda_R$. Straightforward computations show that

$$(3.53) \quad p(\langle E \rangle_{\mathbb{Z}}) \cap \Lambda_R = \langle \alpha_1, \alpha' \alpha''_1, \gamma \rangle_{\mathbb{Z}}$$

and that

$$\begin{aligned}\alpha_1 &= \lambda_1 + \lambda_3 - \lambda_2 - \lambda_4; \\ \alpha' &= \lambda_1 + \lambda_2 - \lambda_3; \\ \alpha''_1 &= \lambda_2 + \lambda_3 - \lambda_1 - \lambda_5; \\ \gamma &= 2\lambda_4 + \lambda_2 - \lambda_1 - \lambda_3 - \lambda_6.\end{aligned}$$

Let K be the basis of $p(\langle E \rangle_{\mathbb{Z}}) \cap \Lambda_R$ given in equation (3.53). Since $\beta \in \langle \Pi \rangle_{\mathbb{N}}$ and all elements of K contain a simple root in their support, which is not in the support of any other element of K , it follows that $\beta \in \langle K \rangle_{\mathbb{N}}$. Therefore, there exist $A_1, A_2, A_3, A_4 \in \mathbb{N}$ such that

$$(3.54) \quad \beta = A_1\alpha_1 + A_2\gamma + A_3\alpha' + A_4\alpha''_1.$$

From the hypothesis of Claim A, it follows that

$$(3.55) \quad \begin{cases} A_1 - A_2 + A_3 - A_4 \leq 0 \\ -A_1 + A_2 + A_3 + A_4 \leq 0 \\ A_1 - A_2 - A_3 + A_4 \leq 0 \end{cases}$$

Adding the last two inequalities in (3.55) yields that $A_4 = 0$. After substituting $A_4 = 0$, the first two inequalities yield that $A_3 = 0$ and then that $A_1 = A_2$. It follows that $\beta \in \langle \alpha_1 + \gamma \rangle_{\mathbb{N}}$. Using that β is the sum of a simple root and an element of $R^+ \cup \{0\}$ (see Proposition 2.6) it follows that $\beta = \alpha_1 + \gamma$ when $m > 2$ and that $\beta \in \{\alpha_1 + \gamma, 2\alpha_1 + 2\gamma\}$ when $m = 2$. With an argument like that for Claim C in the proof of Proposition 3.18, one shows that $2\alpha_1 + 2\gamma$ cannot occur as a T_{ad} -weight in $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ when $m = 2$. This proves Claim A.

We now proceed to Claim B. We fix $\beta = \alpha_1 + \gamma$. We will show that the T_{ad} -weight β has multiplicity at most one in $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$. Since $T_{X_0}M_S^G \subseteq (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$, this implies Claim B. First off, we note that the T_{ad} -weight β only occurs in $V(\lambda_1)$, $V(\lambda_3)$ and in $V(\lambda_4)$, since β belongs to the root lattice of $\text{Sp}(2n)$. Let Z be the subspace of $V(\lambda_1) \oplus V(\lambda_3) \oplus V(\lambda_4)$ consisting of T_{ad} -eigenvectors v of T_{ad} -weight β that satisfy the following three conditions:

$$(3.56) \quad X_{\alpha_1} \cdot v \in \langle X_{-(\beta-\alpha_1)}x_0 \rangle_{\mathbb{C}};$$

$$(3.57) \quad X_{\alpha_2} \cdot v \in \langle X_{-(\beta-\alpha_2)}x_0 \rangle_{\mathbb{C}}; \text{ and}$$

$$(3.58) \quad X_{\alpha_k} \cdot v = 0 \text{ for all } k \in \{1, 2, \dots, n\} \setminus \{1, 2\}.$$

By Proposition 2.6, every $v \in V^{\beta}$ with $0 \neq [v] \in (V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ satisfies (3.56), (3.57) and (3.58). To show Claim B it is therefore enough to prove that

$$(3.59) \quad \dim Z \leq 2,$$

since the nonzero vector $X_{-\beta} \cdot x_0$ belongs to $\mathfrak{g} \cdot x_0 \cap Z$. The proof of (3.59) is the same as that of (3.42). \square

3.15. **(K23) The modules** $((\mathrm{Sp}(2m) \times \mathbb{C}^\times) \times \mathrm{SL}(2) \times (\mathrm{Sp}(2n) \times \mathbb{C}^\times), (\mathbb{C}^{2m} \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{2n}))$ **with** $2 \leq m, n$. For these modules,

$$E = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7\};$$

$$d_W = 5,$$

where

$$\begin{aligned} \lambda_1 &:= \omega_1 + \varepsilon + \omega'; & \lambda_2 &:= \omega' + \omega_1'' + \varepsilon'; & \lambda_3 &:= \omega_1 + \varepsilon + \omega_1'' + \varepsilon'; & \lambda_4 &:= \omega_2 + 2\varepsilon; \\ \lambda_5 &:= \omega_2'' + 2\varepsilon'; & \lambda_6 &:= 2\varepsilon; & \lambda_7 &:= 2\varepsilon'. \end{aligned}$$

We remark that $G = \overline{G}$ is the only connected group between \overline{G}' and \overline{G} for which these modules are spherical, cf. Remark 3.3. Therefore, we assume $G = \overline{G} = (\mathrm{Sp}(2m) \times \mathbb{C}^\times) \times \mathrm{SL}(2) \times (\mathrm{Sp}(2n) \times \mathbb{C}^\times)$ throughout this section.

In this section we will prove the following proposition.

Proposition 3.22. *The T_{ad} -module $T_{X_0}M_S^G$ is multiplicity-free. Its T_{ad} -weight set is*

$$(3.60) \quad \{\alpha_1, \alpha_1 + \gamma, \alpha', \alpha_1'', \alpha_1'' + \gamma''\},$$

where

$$\gamma = \begin{cases} \alpha_2 & \text{if } m = 2; \\ 2(\alpha_2 + \alpha_3 + \dots + \alpha_{m-1}) + \alpha_m & \text{if } m > 2; \end{cases}$$

$$\gamma'' = \begin{cases} \alpha_2'' & \text{if } n = 2; \\ 2(\alpha_2'' + \alpha_3'' + \dots + \alpha_{n-1}'') + \alpha_n'' & \text{if } n > 2. \end{cases}$$

In particular, $\dim T_{X_0}M_S^G = d_W$.

Proof. The argument is very similar to that of Propositions 3.8, 3.20 and 3.21. Let β be a T_{ad} -weight in $T_{X_0}M_S^G$. It follows from Proposition 2.19 that at least one $\lambda \in E$ has a positive coefficient in the expression of β as a \mathbb{Z} -linear combination of elements of E . Note that all elements of E except λ_4 and λ_5 have codimension 1. In particular, if $\lambda \notin \{\lambda_4, \lambda_5\}$, then it follows from Proposition 2.21 that β is a simple root belonging to the set (3.52), and that its weight space has dimension one.

Consequently, to prove the proposition, what remains is to show the following two claims:

Claim A: if λ_4 or λ_5 are the only element of E which have a positive coefficient in the expression of β , then $\beta = \alpha_1 + \gamma$ or $\beta = \alpha_1'' + \gamma''$.

Claim B: the T_{ad} -weights $\beta = \alpha_1 + \gamma$ and $\beta'' = \alpha_1'' + \gamma''$ have multiplicity at most one in $T_{X_0}M_S^G$.

We begin with Claim A. Recall from Proposition 2.6 that $\beta \in \langle E \rangle_{\mathbb{Z}} \cap \langle \Pi \rangle_{\mathbb{N}}$. Straightforward computations show that

$$(3.61) \quad \langle E \rangle_{\mathbb{Z}} \cap \Lambda_R = \langle \alpha_1, \gamma, \alpha', \alpha_1'', \gamma'' \rangle_{\mathbb{Z}}$$

and that

$$\begin{aligned}\alpha_1 &= \lambda_1 + \lambda_3 - \lambda_2 - \lambda_4; \\ \alpha' &= \lambda_1 + \lambda_2 - \lambda_3; \\ \alpha''_1 &= \lambda_2 + \lambda_3 - \lambda_1 - \lambda_5; \\ \gamma &= 2\lambda_4 + \lambda_2 - \lambda_1 - \lambda_3 - \lambda_6; \\ \gamma'' &= 2\lambda_5 + \lambda_1 - \lambda_2 - \lambda_3 - \lambda_7.\end{aligned}$$

Let K be the basis of $\langle E \rangle_{\mathbb{Z}} \cap \Lambda_R$ given in equation (3.61). Since $\beta \in \langle \Pi \rangle_{\mathbb{N}}$ and all elements of K contain a simple root in their support, which is not in the support of any other element of K , it follows that $\beta \in \langle K \rangle_{\mathbb{N}}$. Therefore, there exist $A_1, A_2, A_3, A_4, A_5 \in \mathbb{N}$ such that

$$(3.62) \quad \beta = A_1\alpha_1 + A_2\gamma + A_3\alpha' + A_4\alpha''_1 + A_5\gamma''.$$

From the hypothesis of Claim A, it follows that

$$(3.63) \quad \begin{cases} A_1 - A_2 + A_3 - A_4 + A_5 \leq 0; \\ -A_1 + A_2 + A_3 + A_4 - A_5 \leq 0; \\ A_1 - A_2 - A_3 + A_4 - A_5 \leq 0. \end{cases}$$

Adding the first two inequalities in (3.63) yields that $A_3 = 0$. Adding the first and the third inequality tells us that $A_1 \leq A_2$, while adding the second and third gives $A_4 \leq A_5$. Moreover, after substituting $A_3 = 0$, the first two inequalities also give us that $A_2 - A_1 = A_5 - A_4$. Put $C := A_2 - A_1 = A_5 - A_4$. Then $C \in \mathbb{N}$ and

$$(3.64) \quad \beta = A_1\alpha_1 + (A_1 + C)\gamma + A_4\alpha''_1 + (A_4 + C)\gamma''$$

$$(3.65) \quad = (A_1 + 2C)\lambda_4 + (A_4 + 2C)\lambda_5 - 2C\lambda_3 - (A_1 + C)\lambda_6 - (A_4 + C)\lambda_7.$$

By Proposition 2.20, it follows from (3.65) that $A_1 + A_4 + 4C \leq 2$. This implies that $C = 0$. The inequality $A_1 + A_4 \leq 2$ has five solutions in $\mathbb{N} \times \mathbb{N}$. This implies that $\beta \in \{\beta_1, \beta_2, \dots, \beta_5\}$ where $\beta_1 = \alpha_1 + \gamma$, $\beta_2 = \alpha''_1 + \gamma''$, $\beta_3 = \alpha_1 + \gamma + \alpha''_1 + \gamma''$, $\beta_4 = 2\alpha_1 + 2\gamma$ and $\beta_5 = 2\alpha''_1 + 2\gamma''$. We cannot have $\beta = \beta_3$ because β_3 does not belong to $\Pi + (R^+ \cup \{0\})$. If $m > 2$, then $\beta \neq \beta_4$ for the same reason. If $m = 2$, then an argument like that for Claim C in the proof of Proposition 3.18 shows that $\beta \neq \beta_4$. If $n > 2$, then $\beta \neq \beta_5$ because $\beta_5 \notin \Pi + (R^+ \cup \{0\})$. If $n = 2$, then $\beta \neq \beta_5$ by an argument like that for Claim C in the proof of Proposition 3.18. This proves Claim A.

The argument for Claim B is the same as that for Claim B in the proof of Proposition 3.20, except that one has to go through it twice: first for $\beta = \alpha_1 + \gamma$ and then for $\beta'' = \alpha''_1 + \gamma''$. This finishes the proof. \square

3.16. (K24) The module $(\text{Spin}(8) \times \mathbb{C}^\times \times \mathbb{C}^\times, \mathbb{C}_+^8 \oplus \mathbb{C}_-^8)$. Here

$$\begin{aligned}E &= \{\omega_3 + \varepsilon, \omega_4 + \varepsilon', \omega_1 + \varepsilon + \varepsilon', 2\varepsilon, 2\varepsilon'\}; \\ d_W &= 3.\end{aligned}$$

Proposition 3.23. *The T_{ad} -module $(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$ is multiplicity-free and has T_{ad} -weight set*

$$\{\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4\}.$$

In particular, $\dim(V/\mathfrak{g} \cdot x_0)^{G'_{x_0}} = d_W$. Consequently, $\dim T_{X_0} M_S^G = d_W$

Proof. Note that $G' = \text{Spin}(8)$. Consider the G' -module $V' := V(\omega_1) \oplus V(\omega_3) \oplus V(\omega_4)$ and its element $x'_0 = v_{\omega_1} + v_{\omega_3} + v_{\omega_4}$. Since $V(2\varepsilon)$ and $V(2\varepsilon')$ are subspaces of $\mathfrak{g} \cdot x_0$, we have that $(V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}} \simeq (V/\mathfrak{g} \cdot x_0)^{G'_{x_0}}$ as T_{ad} -modules. Observe that $G'_{x_0} = G'_{x'_0}$.

The monoid $p(\langle E \rangle_{\mathbb{N}}) = \langle \omega_1, \omega_3, \omega_4 \rangle_{\mathbb{N}}$ is free and G' -saturated. By [BCF08, Theorems 3.1 and 3.10], $(V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}}$ is multiplicity-free and its T_{ad} -weights belong to Table 1 in [BCF08, page 2810]. By Proposition 2.6, the T_{ad} -weights of $(V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}}$ also belong to $\Lambda_R \cap \langle \omega_1, \omega_3, \omega_4 \rangle_{\mathbb{Z}}$. A straightforward computation shows that

$$(3.66) \quad \Lambda_R \cap \langle \omega_1, \omega_3, \omega_4 \rangle_{\mathbb{Z}} = \{a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 \mid a, b, c, d \in \mathbb{Z} \text{ and } 2b = a + c + d\}.$$

Let γ be a T_{ad} -weight of $(V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}}$. It follows from equation (3.66) that

$$(3.67) \quad |\text{supp}(\gamma)| \geq 2 \text{ and } \alpha_2 \in \text{supp}(\gamma).$$

There are twelve T_{ad} -weights in [BCF08, Table 1, page 2810] that satisfy (3.67). Six of them are

$$\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4, \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_4, 2\alpha_2 + \alpha_3 + \alpha_4,$$

but γ cannot be among these since they do not belong to $\Lambda_R \cap \langle \omega_1, \omega_3, \omega_4 \rangle_{\mathbb{Z}}$ by equation (3.66).

Three more T_{ad} -weights in [BCF08, Table 1] that satisfy (3.67) are

$$\gamma_1 := 2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 2\omega_1$$

$$\gamma_3 := \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 = 2\omega_3$$

$$\gamma_4 := \alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4 = 2\omega_4$$

We claim that none of them is a T_{ad} -weight in $(V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}}$. Let $i \in \{1, 3, 4\}$. We will argue by contradiction that γ_i is not a T_{ad} -weight in $(V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}}$. Assume $v \in V'$ is a T_{ad} -eigenvector of weight γ_i such that $[v]$ is nonzero in $(V'/\mathfrak{g}' \cdot x'_0)^{G'_{x_0}}$. Note that α_i is the only simple root β such that $\gamma_i - \beta$ is in $R^+ \cup \{0\}$. By Proposition 2.6 this implies that

$$(3.68) \quad X_{\alpha_i} \cdot v \in \langle X_{-(\gamma_i - \alpha_i)} x'_0 \rangle_{\mathbb{C}} \setminus \{0\}.$$

Because $\langle (\gamma_i - \alpha_i)^\vee, \cdot \rangle$ is nonzero on ω_1, ω_3 and ω_4 , we have that $X_{-(\gamma_i - \alpha_i)} x'_0$ has nonzero projection on the three summands $V(\omega_1)$, $V(\omega_3)$ and $V(\omega_4)$ of V' . On the other hand, one checks with *LiE* that γ_i does not occur as a T_{ad} -weight in all three components of V' (see the proof of Proposition 3.15 for the code of a similar computation in *LiE*). This contradiction with equation (3.68) proves the claim.

The remaining three T_{ad} -weights in [BCF08, Table 1] that satisfy equation (3.67) are the three weights listed in the proposition. Since $d_W = 3$ this proves the proposition. \square

APPENDIX: COMPUTING THE INVARIANTS OF THE MODULES IN THE FAMILY K5

During the work for the present paper, we also developed a different technique which explicitly computes the invariants in $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$. The main idea is to use theoretical and elementary arguments to reduce the problem to the study of the smallest case for the parameter n (or the parameters (n, m)), and then do a direct computation for the smallest

case. In this appendix we present the method for the K5 family. The method works equally well for the study of the remaining infinite families.

3.17. Notation and Generalities about Sp_{2n} . To accommodate the computational and explicit nature of this appendix, the notation used here is different from that used in the rest of the paper. Assume $n \geq 1$ is a positive integer. Consider the vector space \mathbb{C}^{2n} with basis e_1, \dots, e_{2n} . We also set $f_i = e_{2n+1-i}$ for $1 \leq i \leq n$.

We define a nondegenerate skewsymmetric bilinear form $\Omega : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \rightarrow \mathbb{C}$ by

$$\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0, \quad \Omega(e_i, f_j) = \delta_{ij}, \quad \Omega(f_i, e_j) = -\delta_{ij}$$

for $1 \leq i, j \leq n$, where δ_{ij} denotes the Kronecker delta function. By definition Sp_{2n} consists of the linear automorphisms g of \mathbb{C}^{2n} which have the property $\Omega(g(v), g(w)) = \Omega(v, w)$ for all $v, w \in \mathbb{C}^{2n}$.

We denote by \mathfrak{sp}_{2n} the Lie algebra of Sp_{2n} . According to [GW09, p.72, Eq (2.8)] it has a basis

$$\{a_{ij}, b_{kl}, c_{kl} : 1 \leq i, j \leq n, 1 \leq k \leq l \leq n\}$$

defined as follows: $a_{ij} = (e_j \mapsto e_i, f_i \mapsto -f_j)$ where the notation means that $a_{ij}(e_j) = e_i, a_{ij}(f_i) = -f_j, a_{ij}(e_t) = 0$ if $1 \leq t \leq n$ and $t \neq j$, and $a_{ij}(f_t) = 0$ if $1 \leq t \leq n$ and $t \neq i$. With the same notational convention $b_{kl} = (e_l \mapsto f_k, e_k \mapsto f_l)$ and $c_{kl} = (f_l \mapsto e_k, f_k \mapsto e_l)$.

3.18. Notation for the K5 example. By definition, for $n \geq 2$, K5 with parameter n , or more simply K5(n), is the K5 family in List (3.1) with group $\mathrm{Sp}_{2n} \times \mathrm{GL}_2$ and $W = \mathbb{C}^{2n} \otimes \mathbb{C}^2$. Fix $n \geq 2$. Set $G = \mathrm{Sp}_{2n} \times \mathrm{GL}_2$.

We denote by $\varepsilon_1, \dots, \varepsilon_n$ the standard basis of the weight lattice of Sp_{2n} and by $\varepsilon'_1, \varepsilon'_2$ the standard basis of the weight lattice of GL_2 . For $1 \leq i \leq n$ we denote by ω_i the i -th fundamental weight of Sp_{2n} , and for $1 \leq i \leq 2$ we denote by ω'_i the i -th fundamental weight of GL_2 . We set:

$$V'_1 = V(\omega_1) \otimes V(\omega'_1), \quad V'_2 = \wedge^2 \mathbb{C}^{2n} \otimes V(\omega'_2), \quad V'_3 = V(\omega'_2), \quad V' = \bigoplus_{i=1}^3 V'_i.$$

We also set

$$V_2 = V(\omega_2) \otimes V(\omega'_2) \subset V'_2, \quad V_i = V'_i \text{ for } i \neq 2, \text{ and } V = \bigoplus_{i=1}^3 V_i.$$

We define, for $1 \leq i \leq 3$, dominant weights λ_i of G by $V_i = V(\lambda_i)$. Hence $\lambda_1 = \omega_1 + \omega'_1$, $\lambda_2 = \omega_2 + \omega'_2$, $\lambda_3 = \omega'_2$. We denote by T the diagonal maximal torus of G . We define an action $\rho' : T \times V' \rightarrow V'$ by $\rho'(t, \sum_{i=1}^3 w_i) = \sum_{i=1}^3 \lambda_i(t) t^{-1} \cdot w_i$, for $t \in T$ and $w_i \in V'_i$ for $1 \leq i \leq 3$. It is clear that for $t \in T$ and $w \in V$ we have $\rho'(t, w) = \alpha(t, w)$, where $\alpha : T \times V \rightarrow V$ is the action defined in [PVS12, Definition 2.11].

For the \mathbb{C}^{2n} that Sp_{2n} acts we fix a basis e_1, \dots, e_{2n} and define $\Omega, f_i, a_{ij}, b_{kl}, c_{kl}$ as in Subsection 3.17. For the \mathbb{C}^2 that GL_2 acts we fix a basis g_1, g_2 and define a basis $\{d_{pq} : 1 \leq p, q \leq 2\}$ of \mathfrak{gl}_2 by $d_{pq}(e_a) = e_p$ if $q = a$ and 0 otherwise. Then the set

$$\{a_{ij}, b_{kl}, c_{kl}, d_{pq} : 1 \leq i, j \leq n, 1 \leq k \leq l \leq n, 1 \leq p, q \leq 2\}$$

is a basis of \mathfrak{g} which we call the standard basis.

We set $\mathcal{A} = \{1, 2, 2n-1, 2n\}$ and $H = \sum_{i=1}^n e_i \wedge f_i \in \wedge^2 \mathbb{C}^{2n}$, $H_s = \sum_{i=1}^2 e_i \wedge f_i \in \wedge^2 \mathbb{C}^2$, $x_0 = e_1 \otimes g_1 + e_1 \wedge e_2 \otimes g_1 \wedge g_2 + g_1 \wedge g_2$. For $1 \leq i \leq 3$ we denote by v_i the component of x_0 that is in V_i . For example, $v_3 = g_1 \wedge g_2$.

We say that $v \in V'$ is an invariant if $[v] \in (V'/\mathfrak{g} \cdot x_0)^{G_{x_0}}$. For $1 \leq i \leq 2n$ we define that the index of e_i is i . Since $f_i = e_{2n+1-i}$, we also define that the index of f_i is $2n+1-i$. We denote by $X_0 \subset V$ the Zariski closure of the G -orbit of x_0 .

We fix the following basis of V' which we call the monomial basis:

$$e_i \otimes g_p : 1 \leq i \leq 2n, 1 \leq p \leq 2, \quad e_i \wedge e_j \otimes g_1 \wedge g_2 : 1 \leq i < j \leq 2n, \quad g_1 \wedge g_2.$$

For $v \in V'$, the terms of v are by definition the nonzero monomial terms of the (unique) expression of v as a linear combination of the monomial basis. For example, the terms of $2e_2 \otimes g_1 + 7g_1 \wedge g_2$ are $2e_2 \otimes g_1$ and $7g_1 \wedge g_2$.

We denote by V'_a , the linear span of the subset of the monomial basis of V' where all indices appearing for e_i are in \mathcal{A} . In more detail, V'_a is the linear span of

$$e_i \otimes g_p : i \in \mathcal{A}, 1 \leq p \leq 2, \quad e_i \wedge e_j \otimes g_1 \wedge g_2 : 1 \leq i < j \leq 2n, \{i, j\} \subset \mathcal{A}, \quad g_1 \wedge g_2.$$

We denote by V'_b the linear span of the remaining elements of the monomial basis of V' .

We define a Lie subalgebra $Z_a \subset \mathfrak{g}$ and a vector subspace $Z_b \subset \mathfrak{g}$ such that, as vector space, \mathfrak{g} is the direct sum of Z_a and Z_b . Namely, we set Z_a to be the linear span of

$$\{a_{ij}, b_{kl}, c_{kl}, d_{ij} : 1 \leq i, j \leq 2, 1 \leq k \leq l \leq 2\}$$

and Z_b to be the linear span of the remaining elements of the standard basis of \mathfrak{g} . Clearly Z_a is in a natural way isomorphic to $\mathfrak{sp}_4 \oplus \mathfrak{gl}_2$ which is the Lie algebra of the group $\mathrm{Sp}_4 \times \mathrm{GL}_2$ of the case K5(2).

An easy direct computation proves the following proposition.

Proposition 3.24. *We have $Z_a \cdot V'_a \subset V'_a$, $Z_a \cdot V'_b \subset V'_b$ and $Z_b \cdot V'_a \subset V'_b$. As a corollary, $Z_a \cdot x_0 \subset V'_a$ and $Z_b \cdot x_0 \subset V'_b$.*

Corollary 3.25. *Assume $v \in \mathfrak{g} \cdot x_0$. Write $v = v_a + v_b$ with $v_a \in V'_a, v_b \in V'_b$. Then there exist $z_a \in Z_a$ and $z_b \in Z_b$ with $v_a = z_a \cdot x_0$ and $v_b = z_b \cdot x_0$.*

Proof. There exists $z \in \mathfrak{g}$ with $v = z \cdot x_0$. Write $z = z_a + z_b$ with $z_a \in Z_a$ and $z_b \in Z_b$, then $v = z_a \cdot x_0 + z_b \cdot x_0$. Using Proposition 3.24 $z_a \cdot x_0 \in V'_a$ and $z_b \cdot x_0 \in V'_b$. Since $V'_a \cap V'_b = \{0\}$ we get $v_a = z_a \cdot x_0$ and $v_b = z_b \cdot x_0$. \square

We denote by \mathfrak{g}_{x_0} the Lie algebra of the stabilizer G_{x_0} of the point x_0 . It will be computed in Proposition 3.28.

Corollary 3.26. *Assume $v \in V'$ is an invariant. Write $v = v_a + v_b$ with $v_a \in V'_a$ and $v_b \in V'_b$. Set $F = Z_a \cap \mathfrak{g}_{x_0}$. Then $F \cdot v_a \subset Z_a \cdot x_0$.*

Proof. If $n = 2$ then $v = v_a$ and the result is obvious. Assume $n \geq 3$ and let $z \in F$. Since $F \subset \mathfrak{g}_{x_0}$ and v is an invariant we have $z \cdot v \in \mathfrak{g} \cdot x_0$. Hence $z \cdot v_a + z \cdot v_b \in \mathfrak{g} \cdot x_0$. Since $z \in Z_a$, by Proposition 3.24 $z \cdot v_a \in V'_a$ and $z \cdot v_b \in V'_b$. Hence $z \cdot v_a + z \cdot v_b$ is the decomposition of $z \cdot v$ with components in V'_a and V'_b . Using Corollary 3.25 we have that $z \cdot v_a \in Z_a \cdot x_0$. \square

Direct computations give the following two propositions.

Proposition 3.27. *The stabilizer subgroup G_{x_0} is equal to the set of $(h_1, h_2) \in \mathrm{Sp}_{2n} \times \mathrm{GL}_2$ which have the property that there exists $b \in \mathbb{C}^*$ and $a_1, a_2 \in \mathbb{C}$ such that*

$$h_1(e_1) = be_1, \quad h_1(e_2) = b^{-1}e_2 + a_1e_1, \quad h_2(g_1) = b^{-1}g_1, \quad h_2(g_2) = bg_2 + a_2g_1.$$

Proposition 3.28. *The following set*

$$\{a_{12}\} \cup \{a_{ij} : 1 \leq i \leq n, 3 \leq j \leq n\} \cup \{b_{ij} : 3 \leq i \leq j \leq n\} \cup \\ \{c_{ij} : 1 \leq i \leq j \leq n\} \cup \{d_{12}\} \cup \{d_{11} - a_{11} - d_{22} + a_{22}\}$$

is a basis for \mathfrak{g}_{x_0} .

Lemma 3.29. *Assume $v \in V'$ is an invariant for $K5(n)$. Write $v = v_a + v_b$ with $v_a \in V'_a$ and $v_b \in V'_b$. Then $[v_a] \in (V'/\mathfrak{g} \cdot x_0)^{\mathfrak{g}_{x_0}}$ for $K5(2)$.*

Proof. Using Proposition 3.28, which implies compatibility of \mathfrak{g}_{x_0} for $K5(n)$ as n varies, the result follows by Corollary 3.26. \square

Proposition 3.30. *Assume $v \in V'$ is a ρ' -weight vector such that $[v] \in (V'/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ and $[v] \neq 0$. Denote by β the ρ' -weight of v . Then β is a linear combination with integer coefficients of the λ_i .*

Proof. Write $v = w_1 + w_2 + w_3$ with $w_j \in V'_j$. Since $v \neq 0$ there exists i with $1 \leq i \leq 3$ such that $w_i \neq 0$. Then the arguments in the proof of [PVS12, Lemma 2.17(c)] also work here, taking into account that by the definition of ρ' we have that w_i is a T -weight vector with T -weight equal to $\lambda_i - \beta$. \square

3.19. Analysis of the invariants of K5 with a high index. Assume $n \geq 2$ and we are in $K5(n)$ case. Proposition 3.34 will give a strong restriction on the invariants v with $v \notin V'_a$. Recall $H = \sum_{i=1}^n e_i \wedge f_i$. We set $\gamma_2 = \varepsilon_1 + \varepsilon_2$ and $q^{(2)} = H \otimes g_1 \wedge g_2$.

Remark 3.31. We have computed G_{x_0} in Proposition 3.27. A small computation shows that $q^{(2)}$ is an invariant.

Lemma 3.32. *Assume $v \in V'$ such that $0 \neq [v] \in (V'/\mathfrak{g} \cdot x_0)^{G_{x_0}}$. Assume v is a ρ' -weight vector. Denote by β the ρ' -weight of v . Assume w is an element of the monomial basis of V' such that $w \in V'_b$ and a nonzero multiple of w is a term of v . Then $n \geq 3, \beta = \gamma_2, w \in V'_2$ and there exists i with $3 \leq i \leq n$ such that $w = e_i \wedge f_i \otimes g_1 \wedge g_2$.*

Proof. Denote by Γ the \mathbb{Z} -span of the weights $\varepsilon_1, \varepsilon_2, \varepsilon'_1, \varepsilon'_2$. We will use that by Proposition 3.30 $\beta \in \Gamma$. The assumption $w \in V'_b$ implies that there exists j with $3 \leq j \leq 2n - 2$ such that e_j appears in w . If $w \in V'_1$ then $w = e_j \otimes g_p$ with $1 \leq p \leq 2$ which implies that $\beta \notin \Gamma$, a contradiction. Hence $w \in V'_2$, so $w = e_p \wedge e_q \otimes g_1 \wedge g_2$ for some $1 \leq p < q \leq n$, or $w = e_p \wedge f_q \otimes g_1 \wedge g_2$ for some $1 \leq p, q \leq n$, or $w = f_p \wedge f_q \otimes g_1 \wedge g_2$ for some $1 \leq p < q \leq n$. The first and the third cases are impossible, since then $\beta \notin \Gamma$. The second case is possible if and only if $p = q$. \square

We need the following lemma, which restricts further the candidate invariants.

Lemma 3.33. *Assume $c_t \in \mathbb{C}$, for $1 \leq t \leq n$. Set $z = (\sum_{t=1}^n c_t e_t \wedge f_t) \otimes g_1 \wedge g_2$. Assume $v \in V'$ is an invariant which is also a ρ' -weight vector with ρ' -weight γ_2 such that, for all $1 \leq t \leq n$ and $d \in \mathbb{C}^*$, we have that $de_t \wedge f_t \otimes g_1 \wedge g_2$ is not a term of $v - z$. Then $c_i = c_j$ for all $3 \leq i < j \leq n$.*

Proof. Fix i, j with $3 \leq i < j \leq n$. We assume $c_i \neq c_j$ and we will get a contradiction. Set $w = e_i \wedge f_j \otimes g_1 \wedge g_2$.

We act by the element $a_{ij} = (e_j \mapsto e_i, f_i \mapsto -f_j)$ which, by Proposition 3.28, is in \mathfrak{g}_{x_0} . We have $a_{ij} \cdot z = (c_i - c_j)w$. Lemma 3.32, which restricts the terms that can appear in v , implies that no nonzero multiple of w can appear as a term of $a_{ij} \cdot (v - z)$. Hence $(c_i - c_j)w$ appears as a term of $a_{ij} \cdot v$. Since v is an invariant, we get that $a_{ij} \cdot v \in \mathfrak{g} \cdot x_0$, hence there exists $z \in \mathfrak{g}$ such that $(c_i - c_j)w$ is a term of $z \cdot x_0$. Since x_0 has all indices less or equal to 2 it follows that each term of $z \cdot x_0$ can have at most one index ≥ 3 . This is a contradiction, since w has two distinct indices, namely $i, 2n + 1 - j$, greater or equal than 3. \square

The following proposition is an important step for the reduction of the problem of invariants for the case K5(n) to the case K5(2).

Proposition 3.34. *Assume $v \in V'$ such that $0 \neq [v] \in (V'/\mathfrak{g} \cdot x_0)^{G_{x_0}}$. Assume v is a ρ' -weight vector. Denote by β the ρ' -weight of v . Write $v = v_a + v_b$ with $v_a \in V'_a$ and $v_b \in V'_b$. Assume $v_b \neq 0$. Then $\beta = \gamma_2$ and there exists $c \in \mathbb{C}^*$ such that $v - cq^{(2)}$ is an invariant contained in V'_a .*

Proof. Lemma 3.32 implies that $\beta = \gamma_2$. Combining Lemmas 3.32 and 3.33 it follows that there exist $c \in \mathbb{C}^*$ such that $v - cq^{(2)} \in V'_a$. By Remark 3.31 $q^{(2)}$ is an invariant, hence $v - cq^{(2)}$ is an invariant contained in V'_a . \square

3.20. The invariants of K5. Recall $H_s = \sum_{i=1}^2 e_i \wedge f_i$ and $\gamma_2 = \varepsilon_1 + \varepsilon_2$. We define the following ρ' -weights and weight vectors in V' : We set $\gamma_1 = \varepsilon_1 - \varepsilon_2 + \varepsilon'_1 - \varepsilon'_2$, and $r_1 = e_2 \otimes g_2$. We set $r_{2,1} = H_s \otimes g_1 \wedge g_2$ and $r_{2,2} = e_1 \wedge f_1 \otimes g_1 \wedge g_2$. Then r_1 is a ρ' -weight vector with ρ' -weight γ_1 and $r_{2,1}$ and $r_{2,2}$ are ρ' -weight vectors with ρ' -weight γ_2 .

An easy direct computation gives the following proposition.

Proposition 3.35. *Assume we are in K5(2). Then $(V'/\mathfrak{g} \cdot x_0)^{G_{x_0}} = (V'/\mathfrak{g} \cdot x_0)^{\mathfrak{g}_{x_0}}$ and the classes in $V'/\mathfrak{g} \cdot x_0$ of the elements*

$$r_1, r_{2,1}, r_{2,2}$$

is a basis for the vector space $(V'/\mathfrak{g} \cdot x_0)^{G_{x_0}}$.

Proposition 3.36. *Assume $n \geq 2$ and we are in K5(n). Then the classes in $V'/\mathfrak{g} \cdot x_0$ of the elements*

$$r_1, q^{(2)}, r_{2,2}$$

is a basis for the vector space $(V'/\mathfrak{g} \cdot x_0)^{G_{x_0}}$.

Proof. If $n = 2$ then the result follows from Proposition 3.35. So we assume that $n \geq 3$.

We have computed G_{x_0} in Proposition 3.27. A small calculation shows that the vectors in the statement of the present proposition are indeed invariants for K5(n).

We will use that if an invariant v for K5(n) is an element of V'_a then, by Lemma 3.29, $[v] \in (V'/\mathfrak{g} \cdot x_0)^{\mathfrak{g}_{x_0}}$ for K5(2), hence by Proposition 3.35 v is an invariant for K5(2).

Assume $v \in V'$ is an invariant which is also a ρ' -weight vector with ρ' -weight β . Write $v = v_a + v_b$ with $v_a \in V'_a$ and $v_b \in V'_b$. We will show that $[v]$ is in the linear subspace of $V'/\mathfrak{g} \cdot x_0$ spanned by the classes of the elements in the statement of the present proposition. Since these classes are linearly independent it will then follow that the classes are a basis for $(V'/\mathfrak{g} \cdot x_0)^{G_{x_0}}$.

We first prove that there exists i with $1 \leq i \leq 2$ such that $\beta = \gamma_i$. If $v_b \neq 0$ it follows from Proposition 3.34 that $\beta = \gamma_2$. If $v_b = 0$ then v is an invariant for K5(2), hence the existence of i such that $\beta = \gamma_i$ follows from Proposition 3.35.

Assume $\beta = \gamma_1$. By Proposition 3.34 $v_b = 0$ and hence v is an invariant for the case K5(2). By Proposition 3.35 the vector subspace of $(V'/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ consisting of vectors with ρ' -weight γ_1 is 1-dimensional spanned by $[r_1]$. Hence $[v]$ is in the linear span of $[r_1]$.

Assume $\beta = \gamma_2$. It follows from Proposition 3.34 that there exists $c_1 \in \mathbb{C}$ such that $v - c_1 q^{(2)}$ is an invariant contained in V'_a . Consequently, by Proposition 3.35 there exist $c_2, c_3 \in \mathbb{C}$ with

$$(3.69) \quad v - c_1 q^{(2)} - (c_2 r_{2,1} + c_3 r_{2,2}) \in \mathfrak{g} \cdot x_0.$$

We will show $c_2 = 0$. Assume $c_2 \neq 0$ and we will get a contradiction. We act by the element $a_{23} = (e_3 \mapsto e_2, f_2 \mapsto -f_3) \in \mathfrak{g}_{x_0}$. Since for all $b \in \mathfrak{g}_{x_0}$ we have $b \cdot (\mathfrak{g} \cdot x_0) \subset (\mathfrak{g} \cdot x_0)$, we get $a_{23} \cdot (\mathfrak{g} \cdot x_0) \subset \mathfrak{g} \cdot x_0$. Since $v, q^{(2)}$ are invariants, we have that $a_{23} \cdot (v - c_1 q^{(2)}) \in \mathfrak{g} \cdot x_0$. Hence, Equation (3.69) implies $a_{23} \cdot (c_2 r_{2,1} + c_3 r_{2,2}) \in \mathfrak{g} \cdot x_0$. Since $r_{2,2}$ is an invariant and $c_2 \neq 0$ it follows that $a_{23} \cdot r_{2,1} \in \mathfrak{g} \cdot x_0$, hence $e_2 \wedge f_3 \otimes g_1 \wedge g_2 \in \mathfrak{g} \cdot x_0$. This is a contradiction, since by Lemma 2.17 (for $t = 1$) the set of nonzero elements of $\mathfrak{g} \cdot x_0$ with ρ' -weight $\varepsilon_2 + \varepsilon_3$ is equal to $\{d(b_{13} \cdot x_0) : d \in \mathbb{C}^*\} = \{d(f_3 \otimes g_1 - e_2 \wedge f_3 \otimes g_1 \wedge g_2) : d \in \mathbb{C}^*\}$. \square

Proposition 3.37. *Assume $n \geq 2$ and we are in K5(n). Then the classes in $V/\mathfrak{g} \cdot x_0$ of the following elements of V*

$$r_1, \quad nr_{2,2} - q^{(2)}$$

is a basis for the vector space $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$.

Proof. By [FH91, Theorem 17.5] $V(\omega_2) \subset \wedge^2 \mathbb{C}^{2n}$ is the kernel of the linear map $\wedge^2 \mathbb{C}^{2n} \rightarrow \mathbb{C}$ uniquely specified by $u_1 \wedge u_2 \mapsto \Omega(u_1, u_2)$ for all $u_1, u_2 \in \mathbb{C}^{2n}$. Hence a basis of $V(\omega_2)$ is

$$e_i \wedge e_j, \quad f_i \wedge f_j, \quad e_a \wedge f_b, \quad e_k \wedge f_k - e_1 \wedge f_1$$

with indices $1 \leq i < j \leq n$, $2 \leq k \leq n$, $1 \leq a, b \leq n$ with $a \neq b$.

Using Proposition 3.36, the result follows by an easy computation. \square

Proposition 3.38. *Assume $n \geq 2$ and we are in K5(n). Then the vector subspace of $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$ consisting of the vectors with the property that the section in $H^0(G \cdot x_0, \mathcal{N}_{X_0})^G$ they induce extends to X_0 is equal to $(V/\mathfrak{g} \cdot x_0)^{G_{x_0}}$.*

Proof. It is enough to show that for each of the 2 basis elements in the statement of Proposition 3.37 the induced section in $H^0(G \cdot x_0, \mathcal{N}_{X_0})^G$ extends to X_0 . Recall $x_0 = \sum_{i=1}^3 v_i$ with $v_i \in V_i$. For $1 \leq i \leq 3$ we set $w_i = x_0 - v_i$. Using [PVS12, Proposition 3.1] we have that, for $i = 1, 2$, the codimension in X_0 of the G -orbit of w_i is ≥ 2 .

We have $\gamma_1 = 2\lambda_1 - \lambda_2$. It follows from Corollary 2.9 that the equivariant section in $H^0(G \cdot x_0, \mathcal{N}_{X_0})^G$ defined by r_1 extends to X_0 .

We have $\gamma_2 = \lambda_2 - \lambda_3$. It follows from Corollary 2.9 that the equivariant section in $H^0(G \cdot x_0, \mathcal{N}_{X_0})^G$ defined by $nr_{2,2} - q^{(2)}$ extends to X_0 . \square

REFERENCES

- [AB05] Valery Alexeev and Michel Brion, *Moduli of affine schemes with reductive group action*, J. Algebraic Geom. **14** (2005), no. 1, 83–117. MR MR2092127 (2006a:14017)
- [ACF14] Roman Avdeev and Stéphanie Cupit-Foutou, *On the irreducible components of moduli schemes for affine spherical varieties*, arXiv:1406.1713v2 [math.AG], 2014.

- [BCF08] P. Bravi and S. Cupit-Foutou, *Equivariant deformations of the affine multicone over a flag variety*, Adv. Math. **217** (2008), no. 6, 2800–2821. MR MR2397467 (2009a:14061)
- [Bou68] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968. MR MR0240238 (39 #1590)
- [BR96] Chal Benson and Gail Ratcliff, *A classification of multiplicity free actions*, J. Algebra **181** (1996), no. 1, 152–186. MR MR1382030 (97c:14046)
- [Bri10] Michel Brion, *Introduction to actions of algebraic groups*, Les cours du CIRM **1** (2010), no. 1, 1–22.
- [Bri13] ———, *Invariant Hilbert schemes*, Handbook of moduli. Vol. I, Adv. Lect. Math. (ALM), vol. 24, Int. Press, Somerville, MA, 2013, pp. 64–117. MR 3184162
- [BVS15] Paolo Bravi and Bart Van Steirteghem, *The moduli scheme of affine spherical varieties with a free weight monoid*, Int. Math. Res. Not. IMRN (2015), 44 pages, advance access, doi: 10.1093/imrn/rnv281.
- [CLS11] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR 2810322 (2012g:14094)
- [FH91] William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics. MR 1153249 (93a:20069)
- [GS] Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at <http://www.math.uiuc.edu/Macaulay2/>.
- [GW09] Roe Goodman and Nolan R. Wallach, *Symmetry, representations, and invariants*, Graduate Texts in Mathematics, vol. 255, Springer, Dordrecht, 2009. MR 2522486 (2011a:20119)
- [JR09] Sébastien Jansou and Nicolas Ressayre, *Invariant deformations of orbit closures in $\mathfrak{sl}(n)$* , Represent. Theory **13** (2009), 50–62. MR MR2485792
- [Kno98] Friedrich Knop, *Some remarks on multiplicity free spaces*, Representation theories and algebraic geometry (Montreal, PQ, 1997), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 514, Kluwer Acad. Publ., Dordrecht, 1998, pp. 301–317. MR MR1653036 (99i:20056)
- [Lea98] Andrew S. Leahy, *A classification of multiplicity free representations*, J. Lie Theory **8** (1998), no. 2, 367–391. MR MR1650378 (2000g:22024)
- [Los09] Ivan V. Losev, *Proof of the Knop conjecture*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 3, 1105–1134. MR MR2543664
- [Lun01] D. Luna, *Variétés sphériques de type A*, Publ. Math. Inst. Hautes Études Sci. (2001), no. 94, 161–226. MR 1896179 (2003f:14056)
- [PVS12] Stavros Argyrios Papadakis and Bart Van Steirteghem, *Equivariant degenerations of spherical modules for groups of type A*, Ann. Inst. Fourier (Grenoble) **62** (2012), no. 5, 1765–1809, extended version at arXiv:1008.0911v3 [math.AG]. MR 3025153
- [Tim11] Dmitry A. Timashev, *Homogeneous spaces and equivariant embeddings*, Encyclopaedia of Mathematical Sciences, vol. 138, Springer, Heidelberg, 2011, Invariant Theory and Algebraic Transformation Groups, 8. MR 2797018 (2012e:14100)
- [TY05] Patrice Tauvel and Rupert W. T. Yu, *Lie algebras and algebraic groups*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2005. MR MR2146652 (2006c:17001)
- [vLCL92] Marc A. A. van Leeuwen, Arjeh M. Cohen, and Bert Lisser, *LiE, a package for Lie group computations*, Computer Algebra Nederland, Amsterdam, 1992, available at <http://www-math.univ-poitiers.fr/~maavl/LiE/>.
- [VP72] È. B. Vinberg and V. L. Popov, *A certain class of quasihomogeneous affine varieties*, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 749–764, English translation in Math. USSR Izv. **6** (1972), 743–758. MR MR0313260 (47 #1815)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOANNINA, 45110 IOANNINA, GREECE
E-mail address: `spapadak@cc.uoi.gr`

DEPARTMENT OF MATHEMATICS, MEDGAR EVERS COLLEGE - CITY UNIVERSITY OF NEW YORK, 1650
BEDFORD AVE., BROOKLYN, NY 11225, USA
E-mail address: `bartvs@mec.cuny.edu`